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EXAMPLE 1 Consider the irrational expressions $A=3+\sqrt{x+2}$ and $B=3-\sqrt{x+2}$, $x \geq -2$. Perform the operations:

- a) A^2 ; b) B^2 ; c) $A \cdot B$.

Solution:

$$\begin{array}{lll}
 \text{a) } A^2 = (3 + \sqrt{x+2})^2 & \text{b) } B^2 = (3 - \sqrt{x+2})^2 & \text{c) } A \cdot B \\
 = 3^2 + 2 \cdot 3\sqrt{x+2} + (\sqrt{x+2})^2 & = 3^2 - 2 \cdot 3\sqrt{x+2} + (\sqrt{x+2})^2 & = (3 + \sqrt{x+2}) \cdot (3 - \sqrt{x+2}) \\
 = 9 + 6\sqrt{x+2} + x + 2 & = 9 - 6\sqrt{x+2} + x + 2 & = 3^2 - (\sqrt{x+2})^2 \\
 = x + 6\sqrt{x+2} + 11 & = x - 6\sqrt{x+2} + 11 & = 9 - (x + 2) \\
 & & = 7 - x
 \end{array}$$

EXAMPLE 2 Simplify the expression $A = \frac{1}{2}(\sqrt{x+1} + \sqrt{x-1})^2$, $x \geq 1$, and calculate its numerical value for $x = \sqrt{10}$.

Solution:

$$\begin{array}{ll}
 \text{1. } A = \frac{1}{2}(\sqrt{x+1} + \sqrt{x-1})^2 & \text{2. For } x = \sqrt{10} \\
 = \frac{1}{2}((\sqrt{x+1})^2 + 2\sqrt{x+1}\sqrt{x-1} + (\sqrt{x-1})^2) & A = \sqrt{10} + \sqrt{(\sqrt{10})^2 - 1} \\
 = \frac{1}{2}(x+1 + 2\sqrt{x^2-1} + x-1) & = \sqrt{10} + \sqrt{10-1} \\
 = \frac{1}{2}(2x + 2\sqrt{x^2-1}) = x + \sqrt{x^2-1} & = 3 + \sqrt{10}.
 \end{array}$$

EXAMPLE 3 Simplify the expression $B = \left(\sqrt{x} - \frac{4\sqrt{x}}{2-\sqrt{x}}\right) \cdot \frac{\sqrt{x}-2}{\sqrt{x}+2}$, $x \geq 0$, $x \neq 4$, and calculate its numerical value for $x = 1\frac{9}{16}$.

Solution:

$$\begin{array}{ll}
 \text{1. } B = \left(\sqrt{x} - \frac{4\sqrt{x}}{2-\sqrt{x}}\right) \cdot \frac{\sqrt{x}-2}{\sqrt{x}+2} & \text{2. For } x = 1\frac{9}{16} \\
 = \frac{2\sqrt{x} - x - 4\sqrt{x}}{2-\sqrt{x}} \cdot \frac{\sqrt{x}-2}{\sqrt{x}+2} & B = \sqrt{\frac{25}{16}} \\
 = \frac{(-x - 2\sqrt{x})(\sqrt{x}-2)}{(2-\sqrt{x})(\sqrt{x}+2)} & = \frac{\sqrt{25}}{\sqrt{16}} = \frac{5}{4} \\
 = \frac{-x\sqrt{x} + 2x - 2x + 4\sqrt{x}}{4-x} & = 1\frac{1}{4}. \\
 = \frac{\sqrt{x}(4-x)}{4-x} = \sqrt{x} &
 \end{array}$$

D₇

When we rewrite a fraction so that there are no radicals in its denominator, we say that we **rationalize the denominator**.

Example: $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{2\sqrt{3}}{(\sqrt{3})^2} = \frac{2}{3}\sqrt{3}$.



To rationalize a fraction with an irrational denominator, we need to multiply the top and the bottom of the fraction by a *suitably chosen factor* so that the product in the new denominator is a rational expression.

Examples: $\frac{9}{\sqrt{3}} = \frac{9\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{9\sqrt{3}}{3} = 3\sqrt{3}$,

$$\frac{1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)} = \frac{\sqrt{2}-1}{(\sqrt{2})^2-1^2} = \frac{\sqrt{2}-1}{2-1} = \sqrt{2}-1.$$

D₈

Two irrational expressions whose product is a rational expression are called **conjugate**.

Examples of conjugate expressions: $2-\sqrt{7}$ and $2+\sqrt{7}$; $\sqrt{5}+\sqrt{2}$ and $\sqrt{5}-\sqrt{2}$.

!

Rule 6: $\frac{a}{\sqrt{b}} = \frac{a\sqrt{b}}{\sqrt{b}\sqrt{b}} = \frac{a\sqrt{b}}{b}, \quad b > 0$

$$\frac{a}{\sqrt{b} \pm \sqrt{c}} = \frac{a(\sqrt{b} \mp \sqrt{c})}{(\sqrt{b} \pm \sqrt{c})(\sqrt{b} \mp \sqrt{c})} = \frac{a(\sqrt{b} \mp \sqrt{c})}{b-c}, \quad b \geq 0, c \geq 0, b \neq c.$$

EXAMPLE 4 Rationalize the denominator of the fraction:

Solution:

a) $\frac{5}{3\sqrt{2}}$;

$$\begin{aligned} \text{a) } \frac{5}{3\sqrt{2}} &= \frac{5\sqrt{2}}{3\sqrt{2}\sqrt{2}} \\ &= \frac{5\sqrt{2}}{3 \cdot 2} = \frac{5\sqrt{2}}{6} \end{aligned}$$

b) $\frac{4}{\sqrt{7}-\sqrt{5}}$;

$$\begin{aligned} \text{b) } \frac{4}{\sqrt{7}-\sqrt{5}} &= \frac{4 \cdot (\sqrt{7}+\sqrt{5})}{(\sqrt{7}-\sqrt{5})(\sqrt{7}+\sqrt{5})} \\ &= \frac{4 \cdot (\sqrt{7}+\sqrt{5})}{7-5} \\ &= 2 \cdot (\sqrt{7}+\sqrt{5}) \end{aligned}$$

c) $\frac{12}{3-\sqrt{5}}$.

$$\begin{aligned} \text{c) } \frac{12}{3-\sqrt{5}} &= \frac{12 \cdot (3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} \\ &= \frac{12 \cdot (3+\sqrt{5})}{9-5} \\ &= 3 \cdot (3+\sqrt{5}) \end{aligned}$$

EXAMPLE 5 Rationalize the denominator of the expression:

Solution: a) $\frac{2a+1}{\sqrt{a}} = \frac{(2a+1)\sqrt{a}}{\sqrt{a} \cdot \sqrt{a}} = \frac{(2a+1)\sqrt{a}}{a}$.

a) $\frac{2a+1}{\sqrt{a}}, a \geq 0$;

$$\begin{aligned} \text{b) } \frac{8}{\sqrt{x}+\sqrt{x-2}} &= \frac{8(\sqrt{x}-\sqrt{x-2})}{(\sqrt{x}+\sqrt{x-2})(\sqrt{x}-\sqrt{x-2})} \\ &= \frac{8(\sqrt{x}-\sqrt{x-2})}{(\sqrt{x})^2-(\sqrt{x-2})^2} \\ &= \frac{8(\sqrt{x}-\sqrt{x-2})}{x-(x-2)} \\ &= 4(\sqrt{x}-\sqrt{x-2}) \end{aligned}$$

b) $\frac{8}{\sqrt{x}+\sqrt{x-2}}, x \geq 2$;

$$\begin{aligned} \text{c) } \frac{3x}{2+\sqrt{x+4}} &= \frac{3x(2-\sqrt{x+4})}{(2+\sqrt{x+4})(2-\sqrt{x+4})} \\ &= \frac{3x(2-\sqrt{x+4})}{2^2-(\sqrt{x+4})^2} \\ &= \frac{3x(2-\sqrt{x+4})}{4-(x+4)} \\ &= 3(\sqrt{x+4}-2) \end{aligned}$$

c) $\frac{3x}{2+\sqrt{x+4}}, x \geq 4, x \neq 0$.

EXERCISES 1. Perform the operations below for the expressions: $A = x + \sqrt{x^2+7}$ and $B = x - \sqrt{x^2+7}$.

a) A^2 ; b) B^2 ; c) $A \cdot B$.

2. Simplify $C = \frac{1}{2}(\sqrt{x+2} - \sqrt{x-2})^2, x \geq 2$, and calculate its value for $x = 2\sqrt{5}$.

112. PROPERTIES OF ARITHMETIC PROGRESSIONS

EXAMPLE 1 Consider the arithmetic progression (+) 5, 8, 11, 14, 17, We notice that:

$$\bullet \quad 8 = \frac{5+11}{2}, \quad 11 = \frac{8+14}{2}, \quad 14 = \frac{11+17}{2};$$

i.e., every term after the first is the arithmetic average of its two neighbors.

T₂

Property 2: A sequence $a_1, a_2, a_3, \dots, a_{n-1}, a_n, a_{n+1}, \dots$ is an arithmetic progression if and only if $a_n = \frac{a_{n-1} + a_{n+1}}{2}$ for all $n \geq 2$.

Proof:

- If a_{n-1} , a_n , and a_{n+1} ($n \geq 2$) are three consecutive terms of an arithmetic progression, from the definition, it follows that

$$d = a_n - a_{n-1} = a_{n+1} - a_n, \quad 2a_n = a_{n-1} + a_{n+1}, \quad a_n = \frac{a_{n-1} + a_{n+1}}{2}.$$

- Conversely, if for any three consecutive terms a_{n-1} , a_n , and a_{n+1} of a sequence the equality $a_n = \frac{a_{n-1} + a_{n+1}}{2}$ is true, it follows that

$$2a_n = a_{n-1} + a_{n+1}; \text{ i.e., } a_n - a_{n-1} = a_{n+1} - a_n \text{ for any } n \geq 2.$$

From here, for $n = 2, 3, 4, \dots$, we obtain:

$$\left| \begin{array}{l} a_2 - a_1 = a_3 - a_2 \\ a_3 - a_2 = a_4 - a_3 \\ a_4 - a_3 = a_5 - a_4 \\ \text{etc.,} \end{array} \right. \Rightarrow a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = a_5 - a_4 = \dots,$$

which shows that the sequence is arithmetic progression.

EXAMPLE 2 For which values of x are the numbers 3, $x + 4$, and $x^2 + 2$ consecutive terms of an increasing arithmetic progression?

Solution: From *Property 2* of arithmetic progressions, we have:

$$\begin{array}{ll} a_n = \frac{a_{n-1} + a_{n+1}}{2} & x + 4 = \frac{3 + x^2 + 2}{2} \\ a_{n-1} = 3 & 2x + 8 = x^2 + 5 \\ a_n = x + 4 & x^2 - 2x - 3 = 0 \\ a_{n+1} = x^2 + 2 & x_1 = 3, \quad x_2 = -1, \\ & (+) 3, 7, 11; \quad (+) 3, 3, 3. \end{array}$$

Answer: For $x = 3$, the numbers 3, $x + 4$, and $x^2 + 2$ are consecutive terms of an increasing arithmetic progression (with $d = 4$).

EXAMPLE 3 For what x are 8, $2x + 1$, and $x^2 - 2x - 1$ consecutive terms of a decreasing arithmetic progression?

Solution:

$$\begin{array}{ll} a_n = \frac{a_{n-1} + a_{n+1}}{2} & 2x + 1 = \frac{8 + x^2 - 2x - 1}{2} \\ a_{n-1} = 8 & 4x + 2 = x^2 - 2x + 7 \\ a_n = 2x + 1 & x^2 - 6x + 5 = 0 \\ a_{n+1} = x^2 - 2x - 1 & x_1 = 1, \quad x_2 = 5, \\ & (+) 8, 3, -2; \quad (+) 8, 11, 14. \end{array}$$

Answer: For $x = 1$, the numbers 8, $2x + 1$, and $x^2 - 2x - 1$ are consecutive terms of a decreasing arithmetic progression (with $d = -5$).

EXAMPLE 4 Consider the arithmetic progression

$$\begin{array}{cccccccccccc} (+) & 1 & , & 2 & , & 3 & , & 4 & , & 5 & , & 6 & , & 7 & , & 8 & , & 9 & , & 10. \\ & \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} & & \underbrace{\hspace{1.5cm}} \end{array}$$

We notice that $1 + 10 = 2 + 9 = 3 + 8 = 4 + 7 = 5 + 6 (= 11)$.

This property works for the terms of *any* finite arithmetic progression.

T₃

Property 3: In any finite arithmetic progression $(+)$ $a_1, a_2, a_3, \dots, a_{n-2}, a_{n-1}, a_n$, the sum of two terms that are at equal distances from the two end terms is equal to the sum of the end terms; i.e., $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \dots$

We will generalize *Property 3*.

For any arithmetic progression (finite or infinite) the sum of any two terms satisfies

$$a_p + a_q = a_1 + (p-1)d + a_1 + (q-1)d = 2a_1 + ((p+q)-2)d,$$

$$a_r + a_s = a_1 + (r-1)d + a_1 + (s-1)d = 2a_1 + ((r+s)-2)d$$

and hence this sum depends only on the sum of the terms' indices. Therefore,

T'₃

Property 3': If two pairs of indices add up to the same sum, then the corresponding pairs of terms of an arithmetic progression also add up to the same sum:

$$p + q = r + s \Rightarrow a_p + a_q = a_r + a_s.$$

In particular, the sums $a_1 + a_n, a_2 + a_{n-1}, a_3 + a_{n-2}, \dots$ are all equal since the corresponding indices also add up to equal sums: $1 + n = 2 + (n-1) = 3 + (n-2) = \dots$

EXAMPLE 5 For an arithmetic progression, we know that $a_4 + a_8 = 42$. Find:

a) a_6 ;

b) $a_3 + a_9$.

Solution: From *Property 3* of arithmetic progressions, we have:

a) $a_4 + a_8 = a_6 + a_6$ ($4 + 8 = 6 + 6$)

b) $a_4 + a_8 = a_3 + a_9$ ($4 + 8 = 3 + 9$)

$$42 = 2a_6$$

$$42 = a_3 + a_9$$

$$a_6 = 21;$$

$$a_3 + a_9 = 42.$$

EXAMPLE 6 For an arithmetic progression, we know that $a_8 = 12$. Find:

a) $a_7 + a_9$;

b) $a_2 + a_{14}$.

Solution:

a) $a_8 + a_8 = a_7 + a_9$ ($8 + 8 = 7 + 9$)

b) $a_8 + a_8 = a_2 + a_{14}$ ($8 + 8 = 2 + 14$)

$$12 + 12 = a_7 + a_9$$

$$12 + 12 = a_2 + a_{14}$$

$$a_7 + a_9 = 24;$$

$$a_2 + a_{14} = 24.$$

EXERCISES For which value of x are the numbers:

1. $x-1, x+3$, and x^2-5 positive consecutive terms of an arithmetic progression?

2. $x-2, x+4$, and x^2-2 consecutive terms of an increasing arithmetic progression?

3. $5, x+5$, and x^2+2 consecutive terms of a decreasing arithmetic progression?

In an arithmetic progression, we know that:

4. $a_7 + a_{13} = 48$. Find: a) a_{10} ; b) $a_5 + a_{15}$.

5. $a_2 + a_{10} = 24$.

Find: a) $a_5 + a_6 + a_7$; b) $a_4 + a_5 + a_6 + a_7 + a_8$.

6. $a_{12} = 64$. Find: a) $a_{10} + a_{14}$; b) $a_6 + a_{18}$.

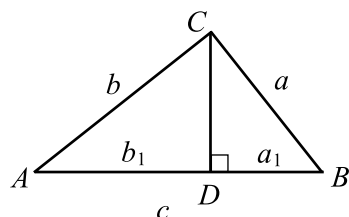
139. THE PYTHAGOREAN THEOREM

T₃

Pythagorean Theorem

The sum of the squares of the legs in a right triangle is equal to the square of the hypotenuse: $a^2 + b^2 = c^2$.

Proof:



We know that $\begin{cases} a^2 = a_1 \cdot c \\ b^2 = b_1 \cdot c \end{cases} +$

We add the two equalities and obtain:

$$\begin{aligned} a^2 + b^2 &= a_1 \cdot c + b_1 \cdot c = \\ &= c \cdot (a_1 + b_1) = c \cdot c = c^2. \end{aligned}$$

The **converse theorem** is also true:

T₄

If the sum of the squares of two sides in a triangle is equal to the square of the third side, then the triangle is right.

The two theorems can be formulated as one in the following way:

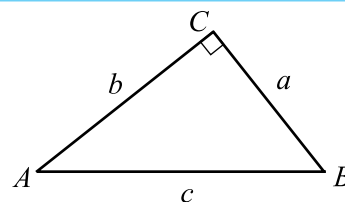
T₅

$\triangle ABC$ with sides $BC = a$, $CA = b$, and $AB = c$ is right with a right angle at vertex C if and only if $a^2 + b^2 = c^2$:

$$\triangle ABC, \angle C = 90^\circ \Leftrightarrow a^2 + b^2 = c^2.$$

EXAMPLE 1 Find the third side of $\triangle ABC$ ($\angle C = 90^\circ$) with legs a and b and hypotenuse c , provided:

- a) $a = 6$ and $b = 8$;
- b) $a = 5$ and $c = 13$;
- c) $b = \sqrt{7}$ and $c = 4$.



Solution:

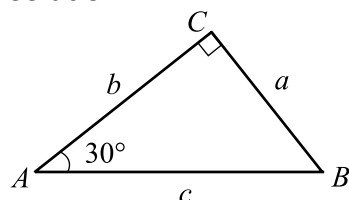
$$\begin{aligned} \text{a) } a^2 + b^2 &= c^2 \\ 6^2 + 8^2 &= c^2 \\ 36 + 64 &= c^2 \\ c^2 &= 100 \\ c &= 10 \end{aligned}$$

$$\begin{aligned} \text{b) } a^2 + b^2 &= c^2 \\ 5^2 + b^2 &= 13^2 \\ 25 + b^2 &= 169 \\ b^2 &= 144 \\ b &= 12 \end{aligned}$$

$$\begin{aligned} \text{c) } a^2 + b^2 &= c^2 \\ a^2 + (\sqrt{7})^2 &= 4^2 \\ a^2 + 7 &= 16 \\ a^2 &= 9 \\ a &= 3 \end{aligned}$$

EXAMPLE 2 Find the legs of a right $\triangle ABC$ ($\angle C = 90^\circ$) with $\angle BAC = 30^\circ$ and $AB = 8$ cm.

Solution:



$$1. a = \frac{c}{2} = \frac{8}{2} = 4 \text{ cm (leg opposite } 30^\circ)$$

$$2. a^2 + b^2 = c^2 \Rightarrow 4^2 + b^2 = 8^2 \Rightarrow$$

$$b^2 = 8^2 - 4^2 = 48 \Rightarrow b = \sqrt{48} = 4\sqrt{3} \text{ cm}$$

EXAMPLE 3 Is $\triangle ABC$ right, provided its sides are:

a) $a = \sqrt{5}$, $b = 2$, $c = 3$;

b) $a = 3$, $b = 4$, $c = 5$;

c) $a = \sqrt{17}$, $b = 2\sqrt{2}$, $c = 5$?

Solution:

$$\begin{aligned} \text{a) } a^2 + b^2 &= \\ &= (\sqrt{5})^2 + 2^2 = \\ &= 5 + 4 = 9 \\ c^2 &= 3^2 = 9 \\ \Rightarrow a^2 + b^2 &= c^2 \\ \Rightarrow \angle C &= 90^\circ \end{aligned}$$

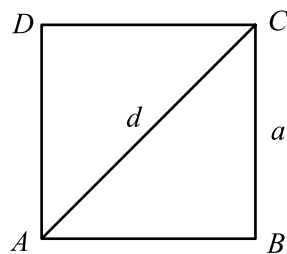
$$\begin{aligned} \text{b) } a^2 + b^2 &= \\ &= 3^2 + 4^2 = \\ &= 9 + 16 = 25 \\ c^2 &= 5^2 = 25 \\ \Rightarrow a^2 + b^2 &= c^2 \\ \Rightarrow \angle C &= 90^\circ \end{aligned}$$

$$\begin{aligned} \text{c) } a^2 + b^2 &= \\ &= (\sqrt{17})^2 + (2\sqrt{2})^2 = \\ &= 17 + 8 = 25 \\ c^2 &= 5^2 = 25 \\ \Rightarrow a^2 + b^2 &= c^2 \\ \Rightarrow \angle C &= 90^\circ \end{aligned}$$

Answer: Yes, $\triangle ABC$ is right.

EXAMPLE 4 Find the diagonal of a square with side a .

Solution:

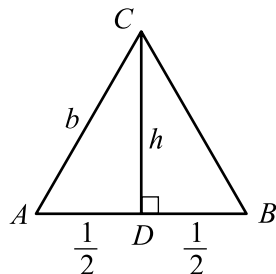


We apply the Pythagorean Theorem to $\triangle ABC$:

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ d^2 &= a^2 + a^2 \\ d^2 &= 2a^2 \\ d &= a\sqrt{2} \end{aligned}$$

EXAMPLE 5 Find the altitude of an equilateral triangle with side b .

Solution:



1. $AD = BD = \frac{b}{2}$ (CD altitude, median)

2. We apply the Pythagorean Theorem to $\triangle ACD$:

$$\begin{aligned} AC^2 &= AD^2 + CD^2 \\ b^2 &= \left(\frac{b}{2}\right)^2 + h^2 \\ h^2 &= b^2 - \frac{b^2}{4} \Rightarrow h^2 = \frac{3b^2}{4} \Rightarrow h = \frac{b\sqrt{3}}{2} \end{aligned}$$

- EXERCISES**
- Find the third side of $\triangle ABC$ ($\angle C = 90^\circ$) with legs a and b and hypotenuse c if:
 - $a = 1$ and $b = 1$;
 - $a = 2$ and $c = 5$;
 - $a = 5$ and $b = 12$;
 - $b = 5$ and $c = 13$.
 - Find the legs of $\triangle ABC$ with $\angle C = 90^\circ$, $\angle BAC = 60^\circ$, and $AB = 10$ cm.
 - Find the legs of an isosceles right triangle with hypotenuse 4 cm.
 - Check if the triangle is right, provided its sides have the following lengths:
 - $a = 15$, $b = 20$, and $c = 25$;
 - $a = 4$, $b = 8$, and $c = 4\sqrt{5}$;
 - $a = 7$, $b = 24$, and $c = 25$;
 - $a = 6$, $b = 8$, and $c = 12$.