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96. RATIONALIZING IRRATIONAL EXPRESSIONS

EXAMPLE 1 Consider the irrational expressions $A=3+\sqrt{x+2}$ and $B=3-\sqrt{x+2}$, $x\geq -2$.
Perform the operations:

a) A^2; b) B^2; c) $A\cdot B$.

Solution:

a) $A^2=\left(3+\sqrt{x+2}\right)^2$
$=3^2+2\cdot 3\sqrt{x+2}+\left(\sqrt{x+2}\right)^2$
$=9+6\sqrt{x+2}+x+2$
$=x+6\sqrt{x+2}+11$

b) $B^2=\left(3-\sqrt{x+2}\right)^2$
$=3^2-2\cdot 3\sqrt{x+2}+\left(\sqrt{x+2}\right)^2$
$=9-6\sqrt{x+2}+x+2$
$=x-6\sqrt{x+2}+11$

c) $A\cdot B$
$=\left(3+\sqrt{x+2}\right)\cdot\left(3-\sqrt{x+2}\right)$
$=3^2-\left(\sqrt{x+2}\right)^2$
$=9-(x+2)$
$=7-x$

EXAMPLE 2 Simplify the expression $A=\frac{1}{2}\left(\sqrt{x+1}+\sqrt{x-1}\right)^2$, $x\geq 1$, and calculate its numerical value for $x=\sqrt{10}$.

Solution:

1. $A=\frac{1}{2}\left(\sqrt{x+1}+\sqrt{x-1}\right)^2$

$=\frac{1}{2}\left((\sqrt{x+1})^2+2\sqrt{x+1}\sqrt{x-1}+(\sqrt{x-1})^2\right)$

$=\frac{1}{2}(x+1+2\sqrt{x^2-1}+x-1)$

$=\frac{1}{2}(2x+2\sqrt{x^2-1})=x+\sqrt{x^2-1}$

2. For $x=\sqrt{10}$

$A=\sqrt{10}+\sqrt{\left(\sqrt{10}\right)^2-1}$

$=\sqrt{10}+\sqrt{10-1}$

$=3+\sqrt{10}.$

EXAMPLE 3 Simplify the expression $B=\left(\sqrt{x}-\frac{4\sqrt{x}}{2-\sqrt{x}}\right)\cdot\frac{\sqrt{x}-2}{\sqrt{x}+2}$, $x\geq 0$, $x\neq 4$, and calculate its numerical value for $x=1\frac{9}{16}$.

Solution:

1. $B=\left(\sqrt{x}-\frac{4\sqrt{x}}{2-\sqrt{x}}\right)\cdot\frac{\sqrt{x}-2}{\sqrt{x}+2}$

$=\frac{2\sqrt{x}-x-4\sqrt{x}}{2-\sqrt{x}}\cdot\frac{\sqrt{x}-2}{\sqrt{x}+2}$

$=\frac{\left(-x-2\sqrt{x}\right)\left(\sqrt{x}-2\right)}{\left(2-\sqrt{x}\right)\left(\sqrt{x}+2\right)}$

$=\frac{-x\sqrt{x}+2x-2x+4\sqrt{x}}{4-x}$

$=\frac{\sqrt{x}(4-x)}{4-x}=\sqrt{x}$

2. For $x=1\frac{9}{16}$

$B=\sqrt{\frac{25}{16}}$

$=\frac{\sqrt{25}}{\sqrt{16}}=\frac{5}{4}$

$=1\frac{1}{4}.$

D7 When we rewrite a fraction so that there are no radicals in its denominator, we say that we **rationalize the denominator.**

Example: $\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}}=\frac{2\sqrt{3}}{\left(\sqrt{3}\right)^2}=\frac{2}{3}\sqrt{3}.$

To rationalize a fraction with an irrational denominator, we need to multiply the top and the bottom of the fraction by a *suitably chosen factor* so that the product in the new denominator is a rational expression.

Examples: $\frac{9}{\sqrt{3}}=\frac{9\sqrt{3}}{\sqrt{3}\sqrt{3}}=\frac{9\sqrt{3}}{3}=3\sqrt{3}$,

$\frac{1}{\sqrt{2}+1}=\frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)}=\frac{\sqrt{2}-1}{(\sqrt{2})^2-1^2}=\frac{\sqrt{2}-1}{2-1}=\sqrt{2}-1.$

D8 Two irrational expressions whose product is a rational expression are called **conjugate.**

Examples of conjugate expressions: $2-\sqrt{7}$ and $2+\sqrt{7}$; $\sqrt{5}+\sqrt{2}$ and $\sqrt{5}-\sqrt{2}$.

!

Rule 6: $\frac{a}{\sqrt{b}}=\frac{a\sqrt{b}}{\sqrt{b}\sqrt{b}}=\frac{a\sqrt{b}}{b},\quad b>0$

$$\frac{a}{\sqrt{b}\pm\sqrt{c}}=\frac{a\left(\sqrt{b}\mp\sqrt{c}\right)}{\left(\sqrt{b}\pm\sqrt{c}\right)\left(\sqrt{b}\mp\sqrt{c}\right)}=\frac{a\left(\sqrt{b}\mp\sqrt{c}\right)}{b-c},\quad b\ge 0,\ c\ge 0,\ b\ne c.$$

EXAMPLE 4 Rationalize the denominator of the fraction:

a) $\frac{5}{3\sqrt{2}}$; b) $\frac{4}{\sqrt{7}-\sqrt{5}}$; c) $\frac{12}{3-\sqrt{5}}$.

Solution:

a) $\frac{5}{3\sqrt{2}}=\frac{5\sqrt{2}}{3\sqrt{2}\sqrt{2}}=\frac{5\sqrt{2}}{3\cdot 2}=\frac{5\sqrt{2}}{6}$

b) $\frac{4}{\sqrt{7}-\sqrt{5}}=\frac{4\cdot\left(\sqrt{7}+\sqrt{5}\right)}{\left(\sqrt{7}-\sqrt{5}\right)\left(\sqrt{7}+\sqrt{5}\right)}=\frac{4\cdot\left(\sqrt{7}+\sqrt{5}\right)}{7-5}=2\cdot\left(\sqrt{7}+\sqrt{5}\right)$

c) $\frac{12}{3-\sqrt{5}}=\frac{12\cdot\left(3+\sqrt{5}\right)}{\left(3-\sqrt{5}\right)\left(3+\sqrt{5}\right)}=\frac{12\cdot\left(3+\sqrt{5}\right)}{9-5}=3\cdot\left(3+\sqrt{5}\right)$

EXAMPLE 5 Rationalize the denominator of the expression:

a) $\frac{2a+1}{\sqrt{a}}$, $a\ge 0$;

b) $\frac{8}{\sqrt{x}+\sqrt{x-2}}$, $x\ge 2$;

c) $\frac{3x}{2+\sqrt{x+4}}$, $x\ge 4$, $x\ne 0$.

Solution: a) $\frac{2a+1}{\sqrt{a}}=\frac{(2a+1)\sqrt{a}}{\sqrt{a}\cdot\sqrt{a}}=\frac{(2a+1)\sqrt{a}}{a}$.

b) $\frac{8}{\sqrt{x}+\sqrt{x-2}}=\frac{8(\sqrt{x}-\sqrt{x-2})}{\left(\sqrt{x}+\sqrt{x-2}\right)\left(\sqrt{x}-\sqrt{x-2}\right)}=\frac{8(\sqrt{x}-\sqrt{x-2})}{\left(\sqrt{x}\right)^2-\left(\sqrt{x-2}\right)^2}=\frac{8(\sqrt{x}-\sqrt{x-2})}{x-(x-2)}=4\left(\sqrt{x}-\sqrt{x-2}\right)$

c) $\frac{3x}{2+\sqrt{x+4}}=\frac{3x(2-\sqrt{x+4})}{\left(2+\sqrt{x+4}\right)\left(2-\sqrt{x+4}\right)}=\frac{3x(2-\sqrt{x+4})}{2^2-\left(\sqrt{x+4}\right)^2}=\frac{3x(2-\sqrt{x+4})}{4-(x+4)}=3\left(\sqrt{x+4}-2\right)$

EXERCISES

1. Perform the operations below for the expressions: $A=x+\sqrt{x^2+7}$ and $B=x-\sqrt{x^2+7}$.

   a) A^2; b) B^2; c) $A\cdot B$.

2. Simplify $C=\frac{1}{2}(\sqrt{x+2}-\sqrt{x-2})^2$, $x\ge 2$, and calculate its value for $x=2\sqrt{5}$.

112. PROPERTIES OF ARITHMETIC PROGRESSIONS

EXAMPLE 1 Consider the arithmetic progression (+) 5, 8, 11, 14, 17, ... We notice that:

- $8=\frac{5+11}{2},\ 11=\frac{8+14}{2},\ 14=\frac{11+17}{2};$

i.e., every term after the first is the arithmetic average of its two neighbors.

T2

Property 2: A sequence $a_1, a_2, a_3, ..., a_{n-1}, a_n, a_{n+1}, ...$ is an arithmetic progression if and only if $\boldsymbol{a_n}=\frac{\boldsymbol{a_{n-1}+a_{n+1}}}{\boldsymbol{2}}$ for all $n \geq 2$.

Proof:

- If a_{n-1}, a_n, and a_{n+1} $(n \geq 2)$ are three consecutive terms of an arithmetic progression, from the definition, it follows that

$$d=a_n-a_{n-1}=a_{n+1}-a_n,\ \ 2a_n=a_{n-1}+a_{n+1},\ \ a_n=\frac{a_{n-1}+a_{n+1}}{2}.$$

- Conversely, if for any three consecutive terms a_{n-1}, a_n, and a_{n+1} of a sequence the equality $a_n=\frac{a_{n-1}+a_{n+1}}{2}$ is true, it follows that

$$2a_n=a_{n-1}+a_{n+1};\text{ i.e., } a_n-a_{n-1}=a_{n+1}-a_n \text{ for any } n \geq 2.$$

From here, for $n = 2, 3, 4, ...,$ we obtain:

$$\left|\begin{array}{l} a_2-a_1=a_3-a_2 \\ a_3-a_2=a_4-a_3 \\ a_4-a_3=a_5-a_4 \\ \text{etc.,} \end{array}\right. \Rightarrow\ a_2-a_1=a_3-a_2=a_4-a_3=a_5-a_4=...,$$

which shows that the sequence is arithmetic progression.

EXAMPLE 2 For which values of x are the numbers 3, $x+4$, and x^2+2 consecutive terms of an increasing arithmetic progression?

Solution: From *Property 2* of arithmetic progressions, we have:

$$a_n=\frac{a_{n-1}+a_{n+1}}{2} \qquad x+4=\frac{3+x^2+2}{2}$$

$$a_{n-1}=3 \qquad 2x+8=x^2+5$$

$$a_n=x+4 \qquad x^2-2x-3=0$$

$$a_{n+1}=x^2+2 \qquad x_1=3, \quad x_2=-1,$$

(+) 3, 7, 11; (+) 3, 3, 3.

Answer: For $x = 3$, the numbers 3, $x + 4$, and $x^2 + 2$ are consecutive terms of an increasing arithmetic progression (with $d = 4$).

EXAMPLE 3 For what x are 8, $2x+1$, and x^2-2x-1 consecutive terms of a decreasing arithmetic progression?

Solution:

$$a_n=\frac{a_{n-1}+a_{n+1}}{2} \qquad 2x+1=\frac{8+x^2-2x-1}{2}$$

$$a_{n-1}=8 \qquad 4x+2=x^2-2x+7$$

$$a_n=2x+1 \qquad x^2-6x+5=0$$

$$a_{n+1}=x^2-2x-1 \qquad x_1=1, \quad x_2=5,$$

(+) 8, 3, –2; (+) 8, 11, 14.

Answer: For $x=1$, the numbers 8, $2x+1$, and x^2-2x-1 are consecutive terms of a decreasing arithmetic progression (with $d=-5$).

EXAMPLE 4 Consider the arithmetic progression

$$(+)\ 1,\quad 2,\quad 3,\quad 4,\quad 5,\quad 6,\quad 7,\quad 8,\quad 9,\quad 10.$$

We notice that $1 + 10 = 2 + 9 = 3 + 8 = 4 + 7 = 5 + 6$ $(= 11)$.

This property works for the terms of *any* finite arithmetic progression.

T3 **Property 3:** In any finite arithmetic progression (+) $a_1, a_2, a_3, \ldots, a_{n-2}, a_{n-1}, a_n$, the sum of two terms that are at equal distances from the two end terms is equal to the sum of the end terms; i.e., $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots.$

We will generalize *Property 3*.

For any arithmetic progression (finite or infinite) the sum of any two terms satisfies

$$a_p + a_q = a_1 + (p-1)d + a_1 + (q-1)d = 2a_1 + ((p+q)-2)d,$$
$$a_r + a_s = a_1 + (r-1)d + a_1 + (s-1)d = 2a_1 + ((r+s)-2)d$$

and hence this sum depends only on the sum of the terms' indices. Therefore,

T'3 **Property 3':** If two pairs of indices add up to the same sum, then the corresponding pairs of terms of an arithmetic progression also add up to the same sum:

$$p + q = r + s \implies a_p + a_q = a_r + a_s.$$

In particular, the sums $a_1 + a_n$, $a_2 + a_{n-1}$, $a_3 + a_{n-2}$, ... are all equal since the corresponding indices also add up to equal sums: $1 + n = 2 + (n-1) = 3 + (n-2) = \ldots.$

EXAMPLE 5 For an arithmetic progression, we know that $a_4 + a_8 = 42$. Find:

a) a_6; b) $a_3 + a_9$.

Solution: From *Property 3* of arithmetic progressions, we have:

a) $a_4 + a_8 = a_6 + a_6$ $(4 + 8 = 6 + 6)$
$42 = 2a_6$
$a_6 = 21$;

b) $a_4 + a_8 = a_3 + a_9$ $(4 + 8 = 3 + 9)$
$42 = a_3 + a_9$
$a_3 + a_9 = 42$.

EXAMPLE 6 For an arithmetic progression, we know that $a_8 = 12$. Find:

a) $a_7 + a_9$; b) $a_2 + a_{14}$.

Solution:

a) $a_8 + a_8 = a_7 + a_9$ $(8 + 8 = 7 + 9)$
$12 + 12 = a_7 + a_9$
$a_7 + a_9 = 24$;

b) $a_8 + a_8 = a_2 + a_{14}$ $(8 + 8 = 2 + 14)$
$12 + 12 = a_2 + a_{14}$
$a_2 + a_{14} = 24$.

EXERCISES

For which value of x are the numbers:

1. $x - 1, x + 3$, and $x^2 - 5$ positive consecutive terms of an arithmetic progression?
2. $x - 2$, $x + 4$, and $x^2 - 2$ consecutive terms of an increasing arithmetic progression?
3. 5, $x + 5$, and $x^2 + 2$ consecutive terms of a decreasing arithmetic progression?

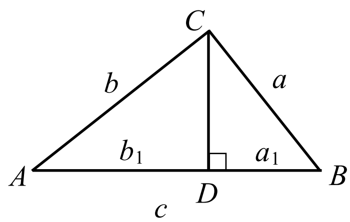
In an arithmetic progression, we know that:

4. $a_7 + a_{13} = 48$. Find: a) a_{10}; b) $a_5 + a_{15}$.
5. $a_2 + a_{10} = 24$.
   Find: a) $a_5 + a_6 + a_7$; b) $a_4 + a_5 + a_6 + a_7 + a_8$.
6. $a_{12} = 64$. Find: a) $a_{10} + a_{14}$; b) $a_6 + a_{18}$.

139. THE PYTHAGOREAN THEOREM

T3 **Pythagorean Theorem**
The sum of the squares of the legs in a right triangle is equal to the square of the hypotenuse: $a^2 + b^2 = c^2$.

Proof:



We know that $\left|\begin{matrix} a^2 = a_1 \cdot c \\ b^2 = b_1 \cdot c \end{matrix}\right\} +$

We add the two equalities and obtain:

$$a^2 + b^2 = a_1 \cdot c + b_1 \cdot c =$$
$$= c \cdot (a_1 + b_1) = c \cdot c = c^2.$$

The **converse theorem** is also true:

T4 If the sum of the squares of two sides is in a triangle is equal to the square of he third side, then the triangle is right.

The two theorems can be formulated as one in the following way:

T5 ΔABC with sides $BC = a$, $CA = b$, and $AB = c$ is right with a right angle at vertex C if and only if $a^2 + b^2 = c^2$:

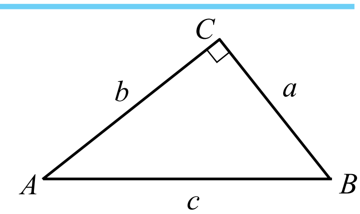
$\Delta ABC, \angle C = 90° \Leftrightarrow a^2 + b^2 = c^2$.

EXAMPLE 1 Find the third side of ΔABC ($\angle C = 90°$) with legs a and b and hypotenuse c, provided:

a) $a = 6$ and $b = 8$;

b) $a = 5$ and $c = 13$;

c) $b = \sqrt{7}$ and $c = 4$.



Solution:

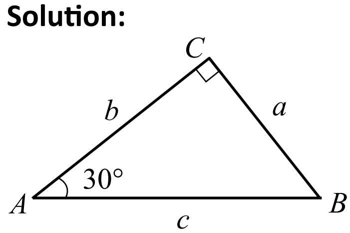
a) $a^2 + b^2 = c^2$
$6^2 + 8^2 = c^2$
$36 + 64 = c^2$
$c^2 = 100$
$c = 10$

b) $a^2 + b^2 = c^2$
$5^2 + b^2 = 13^2$
$25 + b^2 = 169$
$b^2 = 144$
$b = 12$

c) $a^2 + b^2 = c^2$
$a^2 + (\sqrt{7})^2 = 4^2$
$a^2 + 7 = 16$
$a^2 = 9$
$a = 3$

EXAMPLE 2 Find the legs of a right ΔABC ($\angle C = 90°$) with $\angle BAC = 30°$ and $AB = 8$ cm.

Solution:



1. $a = \frac{c}{2} = \frac{8}{2} = 4$ cm (leg opposite 30°)

2. $a^2 + b^2 = c^2 \Rightarrow 4^2 + b^2 = 8^2 \Rightarrow$
$b^2 = 8^2 - 4^2 = 48 \Rightarrow b = \sqrt{48} = 4\sqrt{3}$ cm

EXAMPLE 3 Is ΔABC right, provided its sides are:

a) $a=\sqrt{5}$, $b=2$, $c=3$; b) $a=3$, $b=4$, $c=5$; c) $a=\sqrt{17}$, $b=2\sqrt{2}$, $c=5$?

Solution:

a) $a^2+b^2=$
$=(\sqrt{5})^2+2^2=$
$=5+4=9$
$c^2=3^2=9$
$\Rightarrow a^2+b^2=c^2$
$\Rightarrow \angle C=90°$

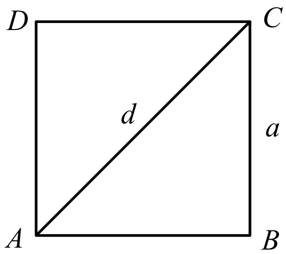
b) $a^2+b^2=$
$=3^2+4^2=$
$=9+16=25$
$c^2=5^2=25$
$\Rightarrow a^2+b^2=c^2$
$\Rightarrow \angle C=90°$

c) $a^2+b^2=$
$=\left(\sqrt{17}\right)^2+\left(2\sqrt{2}\right)^2=$
$=17+8=25$
$c^2=5^2=25$
$\Rightarrow a^2+b^2=c^2$
$\Rightarrow \angle C=90°$

Answer: Yes, ΔABC is right.

EXAMPLE 4 Find the diagonal of a square with side a.

Solution:

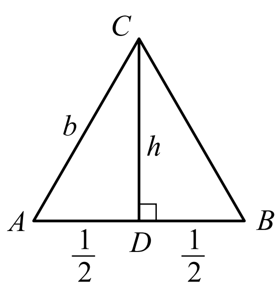


We apply the Pythagorean Theorem to ΔABC:

$AC^2=AB^2+BC^2$
$d^2=a^2+a^2$
$d^2=2a^2$
$d=a\sqrt{2}$

EXAMPLE 5 Find the altitude of an equilateral triangle with side b.

Solution:



1. $AD=BD=\frac{b}{2}$ (CD altitude, median)

2. We apply the Pythagorean Theorem to ΔACD:

$AC^2=AD^2+CD^2$

$b^2=\left(\frac{b}{2}\right)^2+h^2$

$h^2=b^2-\frac{b^2}{4}\Rightarrow h^2=\frac{3b^2}{4}\Rightarrow h=\frac{b\sqrt{3}}{2}$

EXERCISES

1. Find the third side of ΔABC ($\angle C=90°$) with legs a and b and hypotenuse c if:

a) $a=1$ and $b=1$;

b) $a=2$ and $c=5$;

c) $a=5$ and $b=12$;

d) $b=5$ and $c=13$.

2. Find the legs of ΔABC with $\angle C=90°$, $\angle BAC=60°$, and $AB=10$ cm.

3. Find the legs of an isosceles right triangle with hypotenuse 4 cm.

4. Check if the triangle is right, provided its sides have the following lengths:

a) $a=15$, $b=20$, and $c=25$;

b) $a=4$, $b=8$, and $c=4\sqrt{5}$;

c) $a=7$, $b=24$, and $c=25$;

d) $a=6$, $b=8$, and $c=12$.