Zdravka Paskaleva, Maya Alashka

BULGARIAN MATH 2

TEXTBOOK

OR 7 B

adapted for U.S. High Schools
and Excellent Middle Schools

Translated, Adapted, and Augmented
by
Zvezdelina Stankova
Founder and Director
of the Berkeley Math Circle

PUBLISHING HOUSE

Icons, Symbols, and Notation Used in the Textbook:

D **Definition**

A **Axiom**

T **Theorem**

Basic fact/example

In this lesson

Summary of lesson

Pay attention! – Clarification to the solution.

Clarification of the basic facts.

Facts beyond the main material.

Historical notes and background.

Exercises are grouped by difficulty:

- **Mandatory: basic proficiency.**
- **Recommended: high proficiency.**
- **Elective: beyond the main material.**

Write	Read
$\Rightarrow$; $\Leftrightarrow$	implies that; is equivalent to
LHS, RHS	left-hand side, right-hand side
$\alpha, \beta, \gamma, \pi, \varphi, \psi$	alpha, beta, gamma, pi, phi, psi
Δ, Σ	upper-case Delta/Sigma (= sum of)
A', A_1	A-prime, A-sub-one
A z a, $A \in a$	Point A lies on line a.
a z A	Line a passes through point A.
$M \equiv N$, $M = N$	M coincides with (is same as) N.
$M \not\equiv N$, $M \neq N$	M does **not** coincide with N.
$OA^{\rightarrow}$, $\angle AOB$, ΔABC	ray OA, angle AOB, triangle ABC
$a \times b = M$, $a \cap b = M$	Lines a and b intersect in pt M.
$a \perp b$; $a \parallel b$	a and b are perpendicular/parallel.
AV	allowable values (domain)
h_a, m_a, l_a (to side a)	altitude, median, angle bisector
s_{AB}	perpendicular bisector of AB
$\Delta ABC \cong \Delta A_1B_1C_1$	ΔABC is congruent to $\Delta A_1B_1C_1$
SAS, ASA, SSS, SSR	Δ congruence criteria: S = side, A = angle, R = right angle
$k(O; r)$	circle k with center O and radius r

BULGARIAN MATH 2 (7B)
Adapted for U.S. High Schools and Excellent Middle Schools

Edition in English, 2017:

Copyeditors: Thomas Rike, math teacher at Oakland High School; Kelli Talaska, instructor at UC Berkeley

ISBN: 978-954-779-237-1

Edition in Bulgarian, 2014:

Referees: Prof. Dosent Todor Zheliaskov, Dosent Mariana Nenova

Printed by Alliance print – Sofia, Bulgaria

Foreword

No two people will agree entirely on all features of the "best" math curriculum, nor how it should be taught. Consequently, you need to read this introduction with an open mind, accepting that there will be differences in opinions and in approaches, knowing that the "error" in how much the outlined philosophy differs from your views is a proof of the *seriousness of the situation with the U.S. math education.*

We all want to improve the math curriculum and the math experience, and the math proficiency, and the ultimate math success of our children. Do we want too much?

We want a **lot**: that is clear. And a **lot** has to be done in order to achieve this goal: a lot of sacrifices, hard work, and compromises between views of various worlds. For the U.S. is a land where cultures mix (or don't mix but co-exist), it is a land of opportunity, and a land of contradictions and extremes.

Our children come first: **before** our self-centered goals for self-improvement and well-being, **before** ourselves. Sacrificing personal comfort and personal self to the benefit of one's students (or one's children) is a sign of a great teacher (and a great parent).

When something powerful comes along and rings true, along with the listeners will come the frightened ones, who will be afraid to change, who will be scared to death to make the leap and try something new in the U.S.: something that has worked for a century elsewhere.

> From the education battlefield:
>
> *"You have done a masterful job of adapting this curriculum to the U.S. reality.*
>
> *The U.S. Education System needs this kind of prodding. The Common Core ideas are heading in the right direction but the instruments of change are too unwieldy and we cannot just stand by and wait to see what will happen.*
>
> *I hope that your efforts will be successful. Thanks for all of your hard work."*
>
> *Tom Rike*
> *Veteran math teacher*
> *Oakland High School*
> *Copyeditor of the Textbook Series*

And such people will raise angry voices of distrust and will try to stamp out anything tender that has not yet taken root. But without such angry voices there will be no reality check, no feel for the seriousness of the situation.

The following introduction addresses a **typical** situation in a U.S. school. Exceptions do exist. There is no point in starting with this textbook series:

- unless you are ready to be completely *honest, goal-oriented, and selfless* as a teacher, a parent, or an administrator;
- unless you are ready – instead of trying to squash something new because it differs from what you are used to – to *embrace a philosophical viewpoint and a practical attitude that have proven tremendously successful elsewhere* and that have been *adapted to U.S. reality and tested on a pilot school in the San Francisco Bay Area.*

Thus, we hope that you take this new math curriculum seriously and enjoy it too.

Mathematics is a powerful universal language. Everyone needs to and can own it to the extent of his/her abilities and potential. Let this new math curriculum be a big step for the students and community at your middle and high school towards reaching this goal.

Zvezdelina Stankova
Ph.D., Harvard University
Translator, Adapter, and Co-Author
Teaching Professor of Mathematics, UC Berkeley
Berkeley Math Circle Founder & Director
Deborah and Franklin Tepper Haimo Award
for Distinguished College/University Teachers
Princeton's Review "300 Best Professors" in the U.S.

Reviews of the Program
by Teachers, Parents, and Educators

The New Program has attracted fans and followers across the U.S., even before Pilot Year 1 was over.

We received many requests for review copies of the curriculum by schools and teachers, and by many parents, especially in the San Francisco Bay Area and in California in general.

Dr. Oleg Gleizer is the Math Department Chair at *Geffen Academy*, a new pilot middle/high school at UCLA (www.GeffenAcademy.ucla.edu). His goal there is *to create and teach a math curriculum that prepares students for the challenges of the digitized era and showcases the infinite beauty of mathematics.* Under his tutelage, Geffen Academy has adopted this textbook series for its math program and plans to start using it in the fall of 2017.

Dr. Oleg Gleizer graduated with honors from Moscow State University and received his Ph.D. from Northwestern University. He and a student of his have created an algorithm that diagnoses Parkinson's disease in its early stages with nearly 100% accuracy. He has worked for a long time as a math instructor at UCLA and as *the Chief Curriculum Developer and Lead Instructor at the Los Angeles Math Circle* (LAMC)*, where he has trained nine multiple state and national champions in all sorts of Math competitions such as Math Kangaroo, American Mathematics Competitions, and others.

"I was educated in Russia, but live, work, and raise my children in the U.S.

To give my children a proper education in math and sciences, I had to leave my regular academic job and start teaching my kids myself – this is how I ended up working for the Los Angeles Math Circle. The reason was that all the U.S. math textbooks and workbooks I saw were much worse than the ones I myself had used in Russia as a child. The 6A-8C textbooks (or Math 1-5 series) by Dr. Zvezdelina Stankova are a clear exception.

- *On the one hand, they preserve the depth and clarity of the Eastern European school of Math education.*
- *On the other hand, they are tailored to fit the needs of the US children coming out of the U.S. 5th grade in its current state. They are based on the thorough knowledge of the US pre-6th grade curriculum, including the Common Core standards.*

I took a very long and thorough look at the books and found nothing I could improve. The books are a golden standard that should be implemented country-wide. I strongly recommend them to any parent, school, and school district that want to give their children a proper education in Mathematics."

Dr Oleg Gleizer

As was quickly recognized by our pilot teachers, the Bulgarian Math Program can be successfully used:

- in middle schools,

as well as, and very effectively,

- in the first 2 years of high schools.

Dr. Norm Prokup joined the faculty at College Preparatory School in Oakland, CA (the preeminent high school in the SF East Bay Area) in 2003. He has taught at the Berkeley Math Circle and the BMC Summer Program, and very notably, he has been the Geometry instructor for the 7A-7B and 8A-8B groups at the "Math Taught the Right Way" Program at UC Berkeley.

Dr. Norm Prokup graduated from Cornell University and received his Ph.D. from the University of Rhode Island at Kingston, studying nonlinear difference equations that model two-stage "healthy-infected" populations, suggested by epidemiologists. He has taught math ever since, working with students from grade 6 through college, and covering a wide range of subjects, from algebra to ordinary differential equations.

"How powerful is the Bulgarian approach to algebra and geometry? It surpasses the traditional U.S. approach in two key ways:

1. Bulgarian Math presents algebra and geometry simultaneously over three years, guaranteeing much better retention than students can get from the traditional sequence: Pre-algebra, Algebra 1, Geometry, and Algebra 2.
2. The Bulgarian approach is more thorough. Although the authors didn't have the American Common Core standards in mind, the texts work exceptionally well with Common Core. Students who learn algebra and geometry from these texts develop superior skills in problem-solving, logic and reasoning, algebraic manipulation, and computation.

I grew up in northern Virginia. I had good teachers, many of whom cared enough about me to get angry when I wasn't doing my best. Growing up, math was my favorite subject, so I always started my math homework first.

- From engineering school, I learned the value of computation, precision, and fluency of execution.
- From mathematics, I learned that to really understand a subject, you must be able to take it apart and then put it back together.

These values resound throughout the Bulgarian curriculum. All my favorite math problems involve pictures, so I consider myself a geometer at heart.

- Solid geometry, vector geometry, plane transformations, and other topics that are often marginalized in an American curriculum play a central role in the Bulgarian texts. Hence, The Bulgarian curriculum is ideal training for future physicists, engineers, and computer scientists.
- Students emerge from this program ready to study analytic geometry and functions. Thus, this curriculum would be a superior choice for students from all grades 6 to 10.

I use the Bulgarian curriculum for my geometry classes at the UCB Program "Math Taught the Right Way". My students, who range from grade 7 to 10, have embraced this curriculum.

- I love teaching from these texts, and I love sharing them with other teachers.
- I eagerly await each chapter as it emerges, newly translated into English, and
- I'll be first in line to buy each new volume as it becomes available.

The miracle of the Bulgarian Math Program is that it works exceptionally well with a group of mature and industrious younger students. Indeed, the purpose of the Bulgarian texts is to provide the best possible training for tomorrow's mathematicians and scientists.

Dr. Norm Prokup

* * *

These textbooks emphasize the logical structures that seem to lack in the U.S. curriculum. Even starting in the 6A-B series, one can see the build up in knowledge - latter sections require results from previous ones and clearly state so - and the cohesiveness of the algebra and geometry, where the two subjects support each other. When used correctly, the books can explain most results in one lesson, using previous lessons. This structure helps students think mathematically. A true mathematician always asks "Why is this true?" when given a result. It is great to see this type of thinking exposed and instilled in students at such a young age.

In the U.S. math is commonly taught by memorizing algorithms without justifications: "It works because I told you so." And everyone knows that this does not sit well with a rebellious teenager. The essence and beauty of math is whether or not an answer to a question can be found. I was taught to question the results when I started 6^{th} grade axiomatic geometry, using the translated French textbooks in Vietnam. And I am ecstatic to see that the present Bulgarian series provides students the same support and opportunity to question and develop their understanding of mathematics.

Harry Mai-Luu, Math Major at UCB
BMC/MTRW assistant and instructor

Here follow reviews of the Bulgarian Math curriculum by two more instructors, Elysee W.-Egolf and Ellen Kulinsky, at the 6th grade Algebra, Geometry, and Problem Solving classes at the "Math Taught the Right Way" (MTRW) Program at UC Berkeley.

I grew up in the U.S. with textbooks ...

... that were rote and unmotivated, with no applications for algebra students, no geometry or number theory for 6th graders, and no guidelines in writing mathematics. The curricula that are popular in the Bay Area at the time of writing have sought to resolve this issue by making the entirety of student education exploratory, without practice or summarization of what is expected – and in both paradigms, the textbook authors take dozens of pages to get to the point.

The current adapted Archimedes curriculum, by contrast, gives examples for students to explore and then guides them to correct conclusions. Algorithms are offered only after clarifying their logic and students are taught to work clearly and thoroughly through both numerical exercises and problems requiring more complex reasoning – and all this while calmly weaning reliance from calculators. It picks up where books like Singapore Math leave off, and offers students of all backgrounds and abilities entry into the deeper understanding they require to master future material.

Because the topics are covered so thoroughly and a workbook is included with extra problems, there is absolutely no need for teachers to create extra worksheets, which means that the teacher can focus on planning lessons and evaluating student work. The Bulgarian Math textbook is more relaxing for both student and teacher to work from.

Elysee Wilson-Egolf
BA in Mathematics, UC Berkeley
Bentley High School Math Teacher
Instructor at MTRW and BMC

These texts are such an improvement from ...

... the typical textbooks used in the American math education system. Rather than merely presenting formulas, these materials focus on their derivation and various applications. The result is remarkable, studying math becomes engaging, an exploration for the students, rather than the tedious memorization and monotonous plug and chug students may be used to in the American math curriculum.

The books contain beautiful explanations and plenty of examples, easing students into challenging problems. And although the books cover a wide range of mathematical subfields (geometry, algebra, etc.) the students are able to see how concepts from different subfields are integrated, as they are in the real world, outside the classroom.

In my personal experience, growing up, I attended a Math Circle, which exposed me to creative, thought-provoking problem-solving approaches, as well as the connections between mathematical subfields, so I knew that the math classes I was taking in middle and high school were not a fair representation of how enjoyable studying math could be. These are the first math books I have come across that bridge the gap and incorporate the depth, clarity, and excitement of a Math Circle, while also tailored to the American math standards.

These materials are the perfect tool for students and math teachers alike to truly experience and share the beauty and joy of studying mathematics.

Ellen Kulinsky
Math Major, UC Berkeley
Instructor at MTRW and BMC

The quotes on this page are from more pilot teachers, parents, and educators. You will come across them in the Introduction chapter, in their original context. Thus, we strongly encourage you to carefully read the Introduction first, before continuing with the actual math material.

"This book has been a tremendous resource for my students, and I wish I had it when I was in school. The lessons are laid out in a very clear manner. The math is approachable or challenging to meet the needs of a child at any level. There are many math books that claim to reach every person. This book successfully does so, and it is one of the best."

Natalya St. Clair, pilot teacher
co-author, "The Art of Mental Calculation"

"I feel that the Bulgarian texts are much more thorough than the American textbooks I am using for 7^{th} grade. The truth is that had there been this year also a translated 7^{th} grade Bulgarian textbook I could use, I would have definitely used it for all of my classes. It is much easier to teach from and saves me work."

George Khasin, pilot teacher

"The Bulgarian curriculum is more rigorous than others I have used as it introduces Algebra in early grades. Variables are integrated into daily lessons so that students become as familiar with manipulating equations with x and y as they do with real numbers. This will pave the way for an easy transition to Algebra and set them up for success in higher grades."

Laurie Gold, pilot teacher

"Our daughter has ... ... learned to love math with the Bulgarian math curriculum. It stimulates not only her technical mathematical abilities but also challenges her to strategically and creatively solve problems. This program keeps her interested and excited."

Eve Maidenberg
6^{th} grade pilot parent

"Thank you so much... ...for this feedback! Our daughter is struggling to catch up. I feel like she is learning more math now in one month than she did all last year. Thank you for your help!"

6^{th} grade pilot parent

"As a teacher and teacher educator, I am familiar with several... ... math programs that have been used throughout the past 20 years in California. The Bulgarian math program is exceptional in the way it supports students' deep understanding of mathematical concepts, develops students' ability to transfer skills to new and unfamiliar mathematical situations, and creates a mathematical language for students to express themselves mathematically, using numbers, drawings, and narratives.

Teaching is sort of like farming. You never know how your crops are going to turn out, but I have noticed with my son that his work has improved, and I really like the way he is talking about math. I can see, and most importantly hear, the way my child interacts with the Bulgarian math program and I am impressed with the strong foundation it is giving him that will surely lead to lifelong competency and the ability to pursue more advanced mathematics."

Jacqueline Regev, 6th grade pilot parent and educator

"This Middle and High School Math Program is deeply aligned with the Common Core Mathematics Standards".

Dr. Phil Daro
One of the Three Principal Writers of the Common Core Mathematics Standards

7B Contents

Let the Math Continue!

Introduction to the Program

1. Mission Impossible

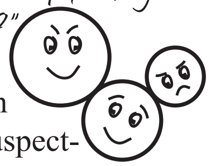
Every year, for the past 20 years, while working* with 250 students in grades 5-12 at the Berkeley Math Circle (BMC), I have been asked over and over again by parents, students, and teachers alike:

"Can you give us good textbooks for our middle and high school math curriculum? What we have is not working and it hasn't worked for years!"

And for these 20 years I could think of only two sources that I would also give to my own children: my *old Bulgarian middle school textbooks*, and the *Art of Problem Solving books (AoPS)*. But **neither of these choices would have worked**, despite the fact that they are, in my opinion, among the best middle school textbooks in the world. Indeed, they are written for *advanced* audiences as measured by U.S. standards:

The very first sentence in my old *Bulgarian 6th Grade Algebra* textbook would stump many of my freshman college students: *"Since we have worked [in 5th grade] with the set of rational numbers* Q *and are familiar with its properties, we move on."* And without further ado, Lesson 9 went on to studying the subtle differences between functions and relations!

My old *Bulgarian 6th Grade Geometry* textbook did not lag behind. On page 42 it asked: *"Is the relation 'is tangent to' on the set of circles in the plane reflexive, symmetric, or transitive?"* Just to explain what the question means to an unsuspecting, typical 6th grader would take a whole lot of time!



Although the *AoPS books* are a wonderful resource, they are not designed as "textbooks" in a regular math class environment. I tend to recommend these books for my more advanced Math Circle students, who are guaranteed to have extra support outside of a typical school classroom.

And so, short of designing a new program myself, I was helpless to help anyone seeking a better middle school math curriculum....

2. The Find

....That is, until I asked my mom to send me some recent Bulgarian math textbooks, to teach my own kids in the U.S. She quickly found the most popular textbooks in Bulgaria: the present series from "Archimedes." When I opened them, I was immediately taken by their main feature: *Each lesson is on exactly 2 pages!*

And each lesson consists of:

- *Several examples of the new concept(s) that illustrate how to understand the new material and how to solve related problems.*
- *A clear summary of new theory, formulae, algorithms, and consequences to be learned.*
- *Up to 10 exercises that differ from each other and that further strengthen the understanding and develop the skills of the reader.*

* I am a Professor of Mathematics at UC Berkeley. I am originally from Bulgaria, which has a rigorous mathematics curriculum. I founded BMC to help pass the math enthusiasm on to the younger generation in the San Francisco Bay Area.

The textbooks are written in a *straightforward* and *simple manner*, capturing the *elegance* and *robustness of mathematics* itself. There is no "hide-and-seek":

- everything is *crystal clear;*
- *one look* at a lesson suffices to see what it will teach you;

- there is no need to look around on the Internet for other sources: the *textbook, workbook,* and *problem collection* are all one needs for a *successful course* in middle school mathematics.

A pilot 6th grade teacher shares:

"I feel that the Bulgarian textbooks are much more thorough than the American textbooks I am using for 7th grade. The truth is that had there been this year also a translated 7th grade Bulgarian textbook I could use, I would have definitely used it for all of my classes. It is much easier to teach from and saves me work."

George Khasin, teacher

3. Origins

This middle school math program is based on one of the leading programs of today in Eastern Europe, developed and implemented by the well-known Bulgarian mathematician and educator Professor Georgi Paskalev. His relatively new mathematical program targeting specifically middle and high schools in Bulgaria has conquered 60-70% of the Bulgarian education market during the single decade since its conception.

The new program:

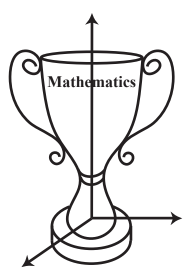
- has *partially departed* from the traditional "old school" Bulgarian program, and
- has *crossed boundaries* to meet the standards and demands of a global educational market, such as in the U.S.; yet,
- has *preserved the intrinsic pedagogical* and *mathematical values* of the old curriculum.

Quite appropriately, the company founded by Professor Paskalev is called "*Archimedes*": its publications live up to the name of the great Greek mathematician, scientist, and inventor from antiquity.*

4. Purpose

The present math program extends the mathematics taught in elementary school and completes it to a comprehensive K-8 math program that prepares students:

- to enter any high school with *confidence* in their math skills and knowledge;
- to be *successful* at any level of high school math classes;



- to develop *maturity* and a *life-long relationship* with mathematics that:
  - will distinguish them among their peers;
  - will serve them well in any profession and in any endeavor in which they chose to engage.

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* Archimedes' picture is from Wikipedia: https://la.wikipedia.org/wiki/Archimedes#/media/File:Archimedes1.jpg.

5. The Temple of Mathematics

The math education of every child is like a *building:* every year we hope that a new floor emerges on top of the previous floor. Unfortunately, more often than not, our children come *under damaging* (even though, perhaps, well-intended) *actions of math educators on any floor of this building*. For example,

- What happens when a teacher decides to spend 3 months reviewing single-digit multiplication in 4th grade?

Well, the 4th graders become really skilled with their multiplication tables, and, say,

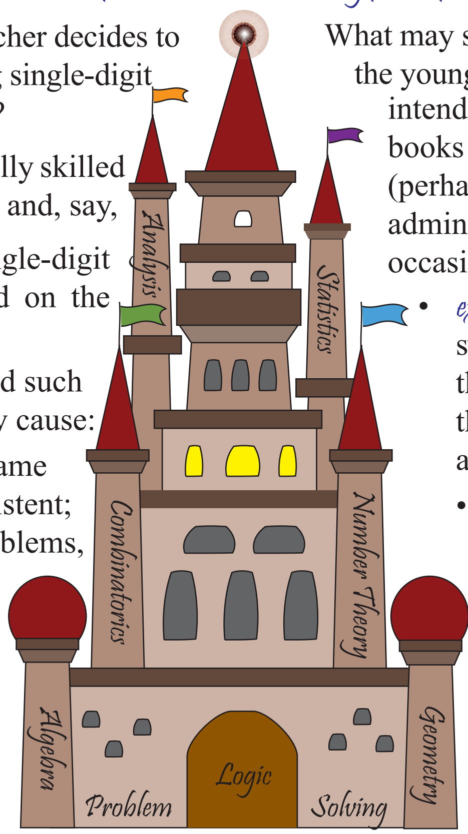
- a 6-foot thick wall of "single-digit multiplication" is erected on the 4th floor of the building.

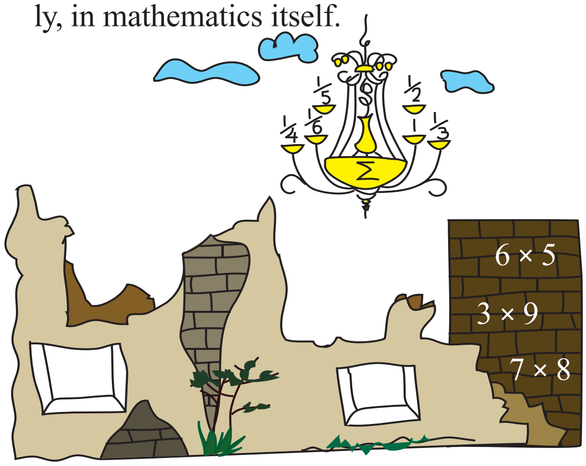
Alas, time is not stretchable, and such educational decisions inevitably cause:

- some other wall on that same floor to be thin or non-existent; e.g., a "wall" of word problems, logical thinking, or a rich geometric vocabulary.

As a result, when in 5th grade the next teacher tries to build "walls" upon such previously expected but non-existent or weak walls, not surprisingly,

- the structure collapses! And in the ensuing debris, time is wasted figuring out what and why it happened. Worse, a student may lose confidence in his/her math skills, in the teacher's abilities, in the integrity of the math program, and, sadly but not infrequently, in mathematics itself.





What may strike us as even *more dangerous* for the young developing mind (yet, still well-intended) is the desire of many textbooks and some teachers nowadays (perhaps, under pressure from school administration or for other reasons) to occasionally show off via:

- *extra-curricular activities* that are unsupported by and incompatible with the math building erected up to that moment. A typical example is an "enrichment" section on:
- *Series*! This topic, ordinarily covered in a college Calculus course, has made its debut in standard, well-known U.S. 6th grade textbooks, alongside computer codes that intend to demonstrate the "growth" of the "partial sums of the series".

Let us reason logically here.

How can one *truly understand* why the sum

$$1 + 1/2 + 1/3 + 1/4 + 1/5 + \cdots$$

adds up to infinity, but the sum

$$1 + 1/2 + 1/4 + 1/8 + 1/16 + \cdots$$

does not, *before* having mastered, say, the *distributivity laws*?! Although U.S. students may be able to recite from memory these distributive laws (e.g., $a \cdot (b + c) = a \cdot b + a \cdot c$), by and large, they will have no clue when, why, and how to apply these laws, so much more efficiently.

Thus, we ask again: what effect will such a "fancy" and untimely introduction of high-level math ideas have on the budding young mind? To put it bluntly, the situation will be equivalent to hanging a fancy chandelier on a non-existent ceiling. Imagine the "aftermath" of such an action, the loss of time, energy, and confidence.

And we are **not** talking about the precocious youngster who can handle with ease and excitement any bombardment with advanced math material, whether on series, in combinatorics, abstract algebra, or other such topics! We have plenty of such young people at the Berkeley Math Circle and observe this phenomenon on a weekly basis all the time.

- We are talking about the *typical* U.S. student, whose primary or only source of learning math is through math classes in school. →
- The math structure for this student will be *jeopardized* by such disorderly and out-of-context teaching.

Just like when learning:

To play the piano, one might first want to learn to read music before attempting *Rachmaninoff's Piano Concerto No. 2;*

To skate, one ought to practice a figure-8 before performing an *Axel jump;*

To write well, one should first concentrate on sentences and paragraphs before producing a literary analysis on, say, "The Realism vs. Local Color in *Mark Twain's novels*"...

... So should there be *order, balance,* and *harmony* in everything we do, teach, and learn. We **cannot** lower our standards when teaching mathematics to the young minds. On the contrary, we must be especially careful because of the *hierarchical structure of mathematics.*

And thus, the present textbook series* has found the *golden ratios*:*

between

- mathematically sound
  and
- pedagogically sound;

between

- the practical
  and
- the inspiring.

The textbook series promotes at all times

Simplicity and Elegance: In anything we do, especially in mathematics.

Courage: Not being afraid to make mistakes and to learn from them.

Diversity of Ideas: Via multiple solutions to the same problem.

First and foremost:
Learning to Think Deeply: Make connections with previously learned material.

These are not just catchy, "politically correct" phrases intended to sell more copies of this textbook. They truthfully describe this math program and help build a solid math structure, with adjoining "walls, windows, and doors," through which to move from topic to topic and make connections between various concepts, formulas, and ideas.

For someone who has not worked deeply with mathematics before, the words *love, creativity, and excitement* are usually light years away from having anything to do with math. Yet, here they all are, in a testimony of a pilot 6th grade parent, leading us to the next topic – *how a student interacts with and feels about mathematics.*

A pilot 6th grade parent shares:

"Our daughter has learned to love math with the Bulgarian math curriculum. It stimulates not only her technical mathematical abilities but also challenges her to strategically and creatively solve problems. This curriculum keeps her interested and excited."

Eve Maidenberg, parent

* The pun was not intended in the former, but it was intended in the latter instance! ☺

6. Relationship with Mathematics

With some notable exceptions, especially in Geometry, the actual math contents in these textbooks does not "look" much different from just about any standard U.S. middle school textbook. After all,

A *linear equation* $5x - 7 = 11$ looks just like a linear equation, no matter if it is displayed in a U.S., Uruguayan, or Bulgarian textbook.

The *Pythagorean Theorem* will still say $a^2 + b^2 = c^2$ for the two legs and the hypotenuse of a right triangle.

A shortcut to finding out if a number can be *divided by 3* (without remainder) will still be to check if the sum of its digits is itself divisible by 3.

... because *mathematical truth is universal*.

Beyond the contents, two *fundamental questions* that a good teacher must ask about any textbook are:

Question 1: How are the math facts *presented?* In what order and what style? Is there a balance between computational (algorithmic) parts and theoretical (abstract) parts?

Question 2: What *relationship* between the student and the material is developed? The deeper the relationship, the more lasting and more useful the math knowledge.

This textbook series creates the environment for a *strong bond between the student and the material,*

by a *gradual climb:*

in the *difficulty and breath* of math content;

in the expectations of *quality, depth, and rigor* of student work;

by building a robust *Temple of Mathematics:*

always *depending on previous knowledge* and reinforcing it via new applications and without needless repetition;

always looking ahead and *linking current material to new things* to discover.

At any time throughout the school year the place and importance of the concepts, ideas, and techniques studied are *clear within the "bigger picture" of Mathematics*. Unfortunately, just as described on the previous pages, meddling with textbooks and curricula is a wide-spread phenomenon in the U.S. middle school math education! Here are four sure ways to ruin the structure and purpose of any math textbook, whether good or bad:

1. Omitting sections.
2. Changing the order of the material in the textbook.
3. Cluttering lessons with handouts from other curricula.
4. Sacrificing the depth and teaching to the answers!...

Remember that no one wants to climb up a flimsy structure! As a result, over time the top floors of the Math Temple will become uninhabitable if the actions in the red box run amok in math lessons.

Instead, the deep thought and math wisdom put into creating these textbooks must be respected, or there is no point in starting with this program. Thus, the present textbooks must be *followed just as systematically* as one studies a new language, violin, or chess.

7. A Cultural Shift

Several features of the textbooks, which are typically ignored or de-emphasized in standard U.S. middle school curricula, represent a *cultural shift* in how mathematics is viewed and studied in the U.S., but which are very well-known around the world and have been adopted by many countries for decades, if not centuries.

7.1. Reading Mathematics. Every math problem has "words" – these are concepts expressed in different ways, whether as numbers, standard everyday words, or by visual aids such as diagrams and figures. Emphasizing the different forms of "words" as a regular part of math language and math communication and *learning to read, interpret, and write in "words"* is a major part of the new curriculum. The pilot parent's comments below addresses this and more:

A pilot 6th grade parent shares, 7 months after the new curriculum was launched:

"As a teacher and teacher educator, I am familiar with several math programs that have been used throughout the past twenty years in California. The Bulgarian math program is exceptional in the way it supports students' deep understanding of mathematical concepts, develops students' ability to transfer skills to new and unfamiliar mathematical situations, and creates a mathematical language for students to express themselves mathematically, using numbers, drawings, and narratives.

I can see, and most importantly hear, the way my child interacts with the Bulgarian math program and I am impressed with the strong foundation it is giving him that will surely lead to lifelong competency and the ability to pursue more advanced mathematics.

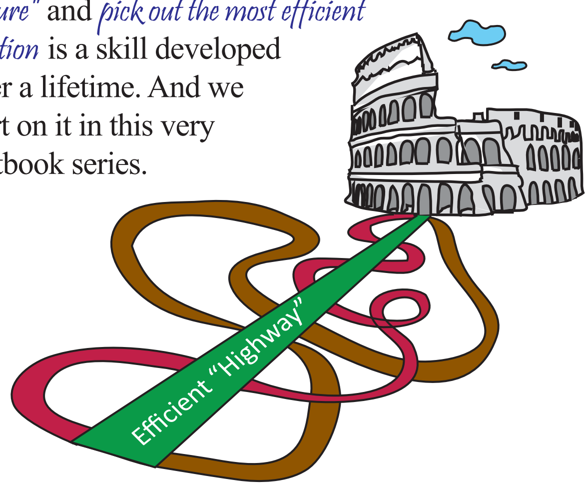
Jacqueline Regev, parent and educator

7.2. Logic and Communication. Learning to know *correct from flawed reasoning* and being *able to explain* one's mathematical ideas smoothly and convincingly to others will represent a gradual move towards *a more mature relationship with mathematics*.

7.3. Writing Mathematics. "Showing your work" is only the beginning to understanding how math really works. All problems have answers in the back of the textbook. Bringing back just an answer on a homework problem from the textbook will be worth no credit. *The explanation that leads to the answer* will be what counts in our study of middle school mathematics. Even harder than learning to correctly interpret problems is learning to consistently *write solutions in a correct, complete, and clear way.* (See Lesson 5 in the 6A textbook.)

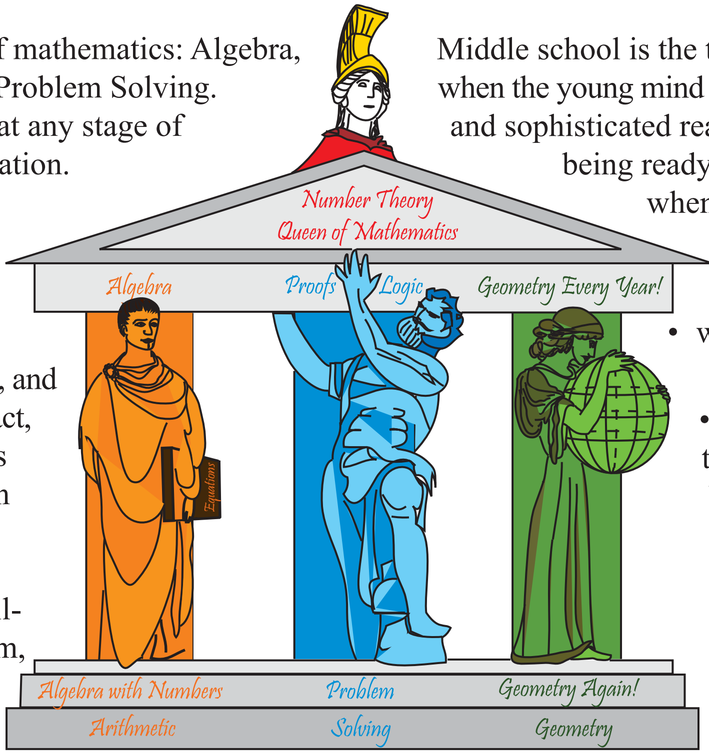
7.4. Multiple Solutions. Although there is usually (but not always!) only *one* correct answer to a math problem, the beauty of mathematics is that there may be *different solutions leading to that answer.* These are not some "subjective opinions"; rather, they are objective mathematical creations that obey the laws of Logic. Students will learn that each solution (and even each incorrect attempt) has its value and usefulness in the long run and that one needs to *open up to others' ideas and ways of thinking* as a path to enriching oneself, to becoming more proficient and, ultimately, wiser.

7.5. Efficient Solutions. Yet, among "all the roads [that] lead to Rome," there might be a shortest or an easiest to follow. The ability to *see "the big picture"* and *pick out the most efficient solution* is a skill developed over a lifetime. And we start on it in this very textbook series.



8. The Pillars of Mathematics

There are *three pillars* of mathematics: Algebra, Geometry, and Logic/Problem Solving. They must be present at any stage of pre-college math education.



8.1. Algebra*

There is so much fuss to distinguish between Pre-Algebra, Algebra 1, and Algebra 2... when, in fact, there is no such thing as "Pre-Algebra" as a math subject, nor are there "Algebra 1 or 2"! In a well-designed and well-executed K-12 program,

- *Arithmetic*

  gradually turns into

- *Algebra* naturally leads to • *Real Analysis.* **

Alas, the typical U.S. student:

- hops along uneven and vaguely defined passages between Pre-Algebra, Algebra 1, Algebra 2, Pre-Calculus, and Calculus;
- experiences a chop-chop approach of poorly matched Algebra curricula from grade to grade;
- is taught to be more concerned about memorizing algorithms*** and passing test benchmarks,

than what is truly important:

- *making connections* and fitting the many pieces together in a giant "jig-saw puzzle" to see the *"big picture"*;

and little by little,

- building a *solid Pillar of Algebra* in his/her Temple of Mathematics.

Middle school is the time when math clicks: when the young mind can handle logical leaps and sophisticated reasoning. Yet, instead of being ready for the "big moment," when middle school knocks on the door, teachers and parents alike start:

- worrying about students *"missing algebra skills"*;
- pulling the class back to arithmetic operations because all of a sudden *"pre-algebra" is too hard;*
- going in circles;
- losing time, energy, and confidence; and
- ultimately, *weakening the math structure.*

8.2. The Missing Algebraic Link

Why is this happening? The major Algebra mishaps in U.S. middle schools are due to events that have already occurred in elementary school, and NOT necessarily due to poor arithmetic skills. Expecting that a middle school student will leap from *arithmetic operations with numbers,* e.g.,

$$97 \cdot 2016 = 195{,}552,$$

to *algebraic operations with symbols,* e.g.,

$$(a + b)\,(a - b) = a^2 - b^2,$$

is asking for trouble. There is a whole array of missing intermediate steps, to which we refer as *"Algebra with Numbers"* and which can and should be started very early in the learning process.

How early? On the next page we list some examples from *early grades in Bulgaria.* They represent well what happens in the algebraic progression also in other countries around the world.

---

* Algebra was present in ancient Greek mathematics mostly as *geometrical* algebra. The word "algebra" is Latin for the Arabic "al-jabr," or "the reunion of broken parts". In our "divine" interpretation of mathematics, the Greek mathematician *Diophantus, the Father of Algebra,* is supporting the math structure on the left and is the only non-deity in the drawing; yet, as a mortal, he had tremendous world influence in both Algebra and Number Theory.

** a.k.a. *Calculus* in first 2 years of U.S. college education. *** Automated "recipes" for some types of problems.

Sample "Algebra with Numbers" Problems from Elementary School in Bulgaria

1st grade. Calculate in two ways $7 + 9 + 3$.

- **Expected written solution:**

$$(7 + 9) + 3 = 16 + 3 = 19;$$
$$(7 + 3) + 9 = 10 + 9 = 19.$$

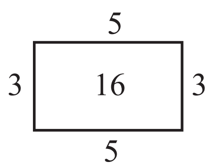
2nd grade. Fill in the blanks. Where have we seen such calculations?

$$\boxed{5 + 5} + \boxed{3 + 3} = 2 \cdot \Diamond + 2 \cdot \Diamond \; = \; \Diamond$$

- **Expected written solution:**

$$\boxed{5 + 5} + \boxed{3 + 3} = 2 \cdot 5 + 2 \cdot 3 \; = \; 10 + 6 = 16.$$

I remember! The sides of a rectangle that are opposite are equal. The perimeter of the rectangle is twice one of the sides plus twice the other side.



3rd grade. Calculate the expression:

$$1000 - (132 \div 3) \cdot 4.$$

- **Expected written solution:**

$$1000 - (132 \div 3) \cdot 4$$
$$= 1000 - 44 \cdot 4 \; = 1000 - 176 = 824.$$

4th grade. Check whether the equality

$$(a + b) - c = a + (b - c)$$

is true for $a = 347$, $b = 217$, and $c = 105$.

- **Expected written solution:**

$$(a + b) - c = (347 + 217) - 105$$
$$= 564 - 105 = 459;$$
$$a + (b - c) = 347 + (217 - 105)$$
$$= 347 + \; 112 = 459.$$

Since we got the same answer 459, the equality is true in this case.

5th grade. Find the unknown number x if

$$8\frac{4}{5} \div x + 5.3 = 25 \cdot 0.3.$$

- **Expected written solution:**

$$8.8 \div x + 5.3 = 7.5$$
$$8.8 \div x = 7.5 - 5.3$$
$$8.8 \div x = 2.2$$
$$x = 8.8 \div 2.2 = 4.$$

This sequence of five random exercises in "Algebra with Numbers" starts in 1st grade in Bulgaria and continues all the way to 6th grade. And then *the "true" Algebra commences,* without hiccups, without drama, and with huge success, because:

- students have been *preparing for Algebra for a long time*, even if no one called it "Algebra";
- a *continuum in studying Algebra* has provided smooth transitions at every step of the way.

And just as playing the piano for the 6th year will likely encompass some harder, lovelier pieces, so will studying Algebra for the 6th year conquer tougher, nicer problems.

This short list of problems, culminating in the 5th-grade equation with fractions and decimals, also explains why we could NOT start our 6th graders in the U.S. on the 6th grade Bulgarian curriculum: they were NOT ready for it! We roughly estimated that they were lagging behind their Bulgarian peers by half a year in terms of algorithmic manipulations; and they were at least a year behind in algebraic skills and math maturity.

Of course, the easiest thing was to give up. But anyone could do that. We could do better. We did MUCH better! In fall 2015 we started the new curriculum in the pilot middle school:

- by first covering all of the Bulgarian 5th grade curriculum in 6th grade here; and
- only then moving to the Bulgarian 6th grade curriculum in the last third of the 6th grade here and covering half of it.

"Were you successful with the Algebra part?" you ask. Let one of the 6th grade teachers respond:

A pilot 6th grade teacher:

"The Bulgarian curriculum is more rigorous than others I have used as it introduces Algebra in early grades. Variables are integrated into daily lessons so that students become as familiar with manipulating equations with x and y as they do with real numbers. This will pave the way for an easy transition to Algebra and set them up for success in higher grades."

Laurie Gold, teacher

8.3. The Geometry Neverland

In our drawing of the pillars of mathematics Geometry is represented by Gaia, the Greek Goddess of Earth ("geo" = Earth in Greek). The truth is that Geometry is the *"Cinderella,"* with U.S. math education being its "step-mom". In K-8 grades Geometry appears sporadically here and there, with few (if any) connections to the rest of the math curriculum. Later, in U.S. high school curriculum, there is *one intense year of Geometry,* during which:

No other math courses are studied. This in itself is a recipe for trouble: the few connections between Geometry and the rest of mathematics that were hinted before in the math curriculum are mostly forgotten during that year.

Geometry is studied in isolation!

4 years worth of Geometry are compressed in one year and they are presented to, by and large, students unprepared for such a huge single "dose" of Geometry.

Unlike making bread, there is no time for Geometry knowledge to "rise"!

Proofs are introduced through this Geometry course, often in the notorious "2-column format". Truly powerful Geometry theorems are not reached. As a result, many students not only do **not** learn how to do proofs, but also...

They end up, by association, disliking Geometry itself!

Question: Why talk about a **high** school Geometry course in the midst of a **middle** school curriculum?

Because everything starts at an early age and builds up over the years, regardless of whether it is Algebra, Geometry, Logic, playing an instrument, or learning to write excellent essays. And if we do not have the *vision of what awaits the young learner* in a few years from now, we will be doing a big disservice to young learner, and we will be cutting off his/her chances to succeed in the future.

Knowledge and skills in Geometry cannot be amassed "miraculously" in a year. *Geometric intuition develops over time,* not instantaneously. To package and present several years worth of Geometry in one course to the untrained mind, and to top it off with introducing formal proofs (a hard topic even in a college curriculum) inevitably leads to a poor outcome.

Worst of all, by the time the high school Geometry course comes, *"the train is already gone":* it is too late to repair the damage that was systematically done over the K-8 years to the Geometry Pillar in the Temple of Mathematics.

Consequently, the young budding math mind has *missed the creative and formative time* in middle school for a fundamental geometric development.

Insufficient Geometry knowledge and skills *will reverberate* throughout a person's whole life, including college and post-college career.

The truth again:
Geometry has been sacrificed in favor of
- Arithmetic in elementary school;
- Algebra in middle school, and
- Pre-Calculus/Calculus in high school.

The *Geometry Pillar will never be built* with the strength and height as it should be in order to hold up a robust and elegant Temple of Mathematics.

The lack of serious and consistent Geometry in early grades makes it impossible to overturn the "algebraic dominance" in the K-12 curriculum. Can this be changed? Can Geometry "Neverland" be replaced by *a healthy symbiosis between Geometry and the rest of mathematics?* Let us look at what could have been.

8.4. A Personal Lamentation

Or … the Golden Geometry Days of My Childhood

From my 25-years of experience teaching just about every undergraduate math course in college, the most mystifying formulas to my students, the least well understood, and (it goes without saying) the least frequently proven in pre-college years are the *trigonometric* ones. During the socialist times *in Bulgaria,* trigonometry was part of the standard 8th grade Geometry curriculum; the theory behind such formulas was systematically studied, and the formulas themselves were rigorously proven via geometric or, as needed, algebraic methods and applied to various problems for years in a row. In fact,

Starting in 6th grade

We had *two separate textbooks:*

- one in Algebra;
- one in Geometry.

We had *separate home-work, tests, and two grades:*

- one in Algebra;
- one in Geometry.

During the week, we *alternated studying:*

- Algebra on MWF;
- Geometry on TTh.

Most importantly, *we had a day between lessons in the same subject,* which allowed us:

to think deeply about what we studied in class;

to try various approaches on the homework problems;

and ultimately, to make the Algebra and the Geometry material *"our own".*

Although in two formally separate subjects, our studies of Algebra and Geometry were synchronized and mutually enriching. *A constant and deep relationship between Geometry and Algebra* was reinforced at all stages of pre-college education: by using geometric facts to supply algebraic reasoning, and by using algebraic formulas to solve Geometry problems.

The Geometry Pillar of Mathematics was gradually built upwards throughout every single grade. There was no week without Geometry, especially from middle school onward. To this day, I distinctly remember the yearly Geometry themes that we studied, starting in 6th grade:

6th grade: *Plane Geometry* start of proofs

7th grade: *Circle Geometry* with proofs

8th grade: *Trigonometry* with proofs

9th grade: *Vector Geometry* with proofs

10th grade: *Solid Geometry* with proofs

11th grade: *Elective Geometry* advanced topic

Former Bulgarian Geometry Curriculum in Middle and High School

Every theme automatically included applications of the Geometry theory and techniques to *problem solving* and contributed to a *balanced and rich math curriculum.*

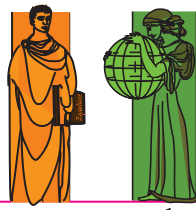
This is the *ideal symbiotic model of Geo-Algebra* that I would like for my children and grandchildren.

Unfortunately, high school math curriculum in the U.S. is driven, by and large, by college requirements and, let's face it, earning "bonus entrance points" to the more competitive universities. Hence, it is ultimately a *Calculus-driven curriculum,* regardless of whether students complete Calculus courses in high school, or not.

In turn, middle school math curriculum aligns itself with high school priorities. Backtracking, the pre-requisite to Calculus is Pre-Calculus, and the pre-requisite to Pre-Calculus is Algebra. Thus, Algebra "wins" by default, due to an established *Algebra-driven* and *Algebra-dominated environment.*

8.5. A Reality Check: Geo-Algebra in the U.S.?!

It is unrealistic (although not entirely impossible) to imagine that the pre-college curriculum in the U.S. can quickly change so much that it will embrace Geometry as an equal counterpart of Algebra, will split textbooks and classes into Algebra and Geometry ones, and will employ *the Geo-Algebra symbiotic model.*



The present textbook series capitalizes on a compromise between Eastern European and U.S. math educational systems and realities.

On the other hand, times **have** changed. *The Bulgarian curriculum has moved part-way towards the U.S. one;* e.g., newer Bulgarian textbooks feature both Algebra and Geometry, while preserving the mathematical integrity and pedagogical insights of a tremendously successful curriculum from the near past.

At a Glance:
The New Middle School Curriculum Adapted to the U.S. Reality by Topic and Semester

6A Textbook

Decimals operations, algebra with numbers; calculate efficiently; motion & word problems

Divisibility criteria for divisibility; GCD and LCM

Plane Geometry altitudes; parallel lines; quadrilateral: perimeter, area; apply algebra

Solid Geometry rectangular parallelepiped: surface area, volume; applications

Write: correct, complete, and clear solutions.

6B Textbook

Fractions, %'s operations; part of a whole; applications

Exponents rules; properties

Rational #'s operations with pos./neg. numbers; modulus; equations

Graphing, Data coordinates, diagrams/tables, arithmetic mean

Plane Geometry circles, discs, polygons: perimeter, area; regular n-gons; constructions

Recognize: expressions and equations.

Split: solutions into "given," "to find," and calculations.

Draw: pictures and diagrams

Math 1 7A Textbook

Proportions properties, applications

Polynomials variables, numerical values, degrees, operations, algebraic identities; factoring in degrees 2 and 3

Solid Geometry types of prisms and pyramids; surface area, volume; constructions, applications

Include: relevant formulas with variables; then substitute given numerical data.

Apply: algebra tools to solve geometry problems.

Math 2 7B Textbook

Equations linear, factored quadratic, parametric; w/ absolute value; modeling, practical applications

Plane Geometry basic notions: pairs of angles, perpendicular/ parallel lines; criteria, axioms, theorems

Congruent Δ's criteria, isosceles Δ, perpendicular/angle bisectors, Euclidean constructions

Understand: the structure of mathematics

Start proofs: using definitions, primary concepts, axioms, and theorems.

Math 3 8A Textbook

Inequalities properties, one variable, linear; representations vs. equations; inequalities in Δ

Square Roots Quadratic Eq'ns irrational numbers; algebraic expressions; quadratic f-la

Inequalities in Δ btw sides, Triangle Inequality

Quadrilaterals parallelogram, trapezoid, rhombus, properties, criteria, theorems, proofs, constructions, applications

Devise: algorithms for geometric constructions;

Link: facts with given info.

Math 4 / 8B Textbook

Functions graphs; directly and inversely proportional; linear and absolute value functions; applications to equations and inequalities; graph of ax^2

Systems of Equations and Inequalities linear w/ 1 or 2 unknowns, w/ modulus; modeling, applications

Transformations translations, rotations, reflections, central symmetries; congruences; definitions, properties; constructions, applications to polygons and circles

Determine: the hypothesis and what to prove;

Construct: a proof as a logical sequence with words and math symbols.

Math 5 8C Textbook

Vectors operations; applications to midsegments and centroids

Circle Geometry tangents; angles in a circle; geometric loci; circumcircles and incircles; famous pts in Δ; circumscribed and inscribed quadrilaterals

Find: multiple ways to solve.

Distinguish: between faulty and valid reasoning.

Learn: from own experience and others' solutions.

Legend

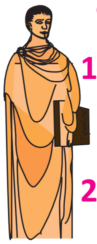
- *Algebra*
- *Geometry*
- *Statistics*
- *Number Theory*
- *Logic and Proofs*

8.6. The Main Principles

The Main Principles in designing this middle school math curriculum are

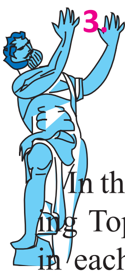
Continuity + Balance + Integrity

of the 4 basic building math blocks:



1. **Algebra** is developed from an early stage, using numbers at first and then more and more frequently variables.
2. **Geometry** is developed alongside Algebra as an equal partner.

Algebra & Geometry support and reinforce each other via applications of each subject into the other.



3. **Logic, Proofs, and Problem Solving*** comprise the skeleton of the Math Temple, providing the "glue" that keeps together all parts of mathematics.

In the syllabus on p. XX, the Logic/Proof-Writing Topics are formally split into two semesters in each grade. In reality, they appear uniformly throughout the whole middle school curriculum.

4A. Number Theory, or the Queen of Mathematics,** makes a grand appearance through the topic of *Divisibility* in the 6A textbook (and later), to aid both logical development and practical algebra skills; e.g., finding efficiently *GCD* and *LCM*.

If you wonder what Number Theory is all about, try to decide if 9,876,543,210 is divisible by 15 *without actually dividing by* 15! The world of *whole* numbers is how Number Theory can be approximately described in middle school.

4B. Statistics and Applications are ubiquitous in the curriculum, aiming at data graphing, optimization, and approximations, and aided by physical experimentation.

9. Differentiate and Scaffold

Talking about differentiation in math classes is a "politically correct" and enticing buzzword to mention when advertising a curriculum and a school. Yet, by and large, this talk is not matched by the little done to truly support both ends of the spectrum... because it is hard to scaffold math lessons and because *it takes an enthusiastic math specialist to successfully cope with diversity in the math classroom.*

When mathematics is taught:

- in an organized and sound way,
- building a solid Math Temple for everyone,

the number of students who "fall through the cracks" and lag behind is minimized. Students who are ahead, though, need something else: something beyond the regular curriculum that will still keep them topic-wise in the same math vicinity as the rest of their classmates. To this end, the exercises in the **Textbooks** are split into:

1 *Basic* (mandatory) problems;
2 3 *High Proficiency* (recommended) problems;
4 *Harder/Math Contest* (elective) problems.

The **Workbooks** provide solid "workouts" on the basic material. While everyone should complete a substantial portion of them, the Workbooks are an absolute must for the weaker students, who should work extensively with them.

In addition, a **Problem Collection for the Teacher** is developed for every grade to match the material studied in every chapter of the textbook. Some problems are grouped in multiple-choice tests, and some are open-ended and organized by topic. The material is gradated in three levels:

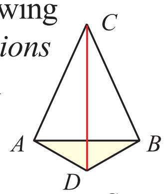
- **Level A:** basic material, building the fundamentals and providing practice for everyone.
- **Level B:** elective material, extending the knowledge for those who want to move forward.
- **Level C:** preparation for math contests for the more adventurous and/or advanced students.

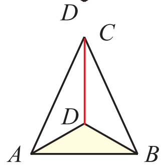
* The role of Logic, Proofs, and Problem Solving is played by the Greek Titan *Atlas,* often pictured "holding the Earth on his shoulders". In our situation, Atlas represents the *central Pillar of Mathematics,* which keeps the whole structure together.

** Number Theory is widely regarded as the Queen of Mathematics. Indeed, some of the most famous and hard math problems have been in Number Theory; e.g., Fermat's Last Theorem. In our Math Temple, Number Theory is played by *Athena,* the Greek Goddess of Mathematics (also of wisdom, courage, inspiration, war strategy, arts, literature, and others). In the textbook series, Athena also supports the areas of Graphing and Data Analysis, or Statistics, and other applications.

9.1. Examples of Scaffolding

The 7B Textbook features the following exercises from the Chapters on *Equations* and *Congruent Triangle* that reach all parts of the difficulty spectrum:





1 (*Basic*). ΔABC and ΔABD with a common base AB are isosceles. Prove that $\Delta ADC \cong \Delta BDC$ and that CD is the angle bisector of the angles at vertices C and D.

2 (*High Proficiency*). A motor boat covers the distance between two ports traveling with the current of the river in 4 h, and coming back against the current in 5 h. What is the distance between the ports if the speed of the current is 2 km/h?

3 (*Hard, but doable w/ school material*). In ΔABC we have $\alpha : \beta : \gamma = 1 : 8 : 3$. The perpendicular bisectors of sides AB and BC intersects side AC correspondingly in points M and N. It is **not** true that:

A) $\angle BNC = 90°$; B) $P_{\Delta MNB} = AC$;
C) $AM : NC = 2 : 1$; D) $AM = BC$.

4 (*More advanced material and ideas*). Construct *three centers* in an acute ΔABC: circumcenter O, orthocenter H, and centroid M. Repeat this for a right and for an obtuse ΔABC. In each of your three pictures confirm that H, M, and O lie on the same line (called *Euler's line*) so that M is between H and O and divides it in ratio $HM : MO = 2 : 1$. Of which other problem does this remind you?

Occasionally, the textbook uses the eye-icon to signify a fact or technique that is *beyond the textbook material.* For example, Lesson 136 on *Geometric Constructions* challenge the reader:

> By using only a ruler and a right angle tool, it is possible to split any segment into three [or any number n] exactly equal parts.

The *Answers* Section itself features an advanced mini-lesson using a version of the famous so-called *Arithmetic-Geometric Mean Inequality*:

> **T** For any $x \geq 0$ and $y \geq 0$ we have
> $$4xy \leq (x+y)^2.$$
> We obtain equality $4xy = (x+y)^2$ if and only $x = y$.

9.2. What is Math 5/8C Textbook?

This U.S.-adapted program starts in 6th grade with the 5th grade Bulgarian material; that is,

- In the beginning, it lags a *full year* behind its Bulgarian counterpart.
- By the end of 6th grade, it lags only *half a year*.
- By the end of 8th grade, the difference is cut down to only a *quarter of a year*.

All of the original 5th-8th grade Algebra material and most of the original Geometry material is covered through the current 6th-8th grade textbooks.

The harder half of the original 8th Geometry material on *Vectors and Circle Geometry* comprises the extra 8C textbook. It is intended *for the more advanced students,* perhaps, to be covered as:

- an elective 8th grade course in Geometry.

The extra material in the 8C textbook is considerably ahead of the standard Geometry material in U.S. high schools and it will be:

- a *superb preparation* for those who would like *to be challenged and to excel in Geometry.*

9.3. Does the Scaffolding Work?

Does the program really benefit all levels of students? Here is some anecdotal evidence.

Answer 1: A month after the start of the pilot 6th grade program, the teacher received a letter from a parent of a student who was struggling tremendously in the beginning, not participating at all in class, and submitting practically empty tests:

> *"Thank you so much for this feedback! She is struggling to catch up. I feel like she is learning more math now in one month than she did all last year. Thank you for your help!"*
>
> *6th grade pilot parent*

The student gradually progressed to very good performance both on tests and in class, showing enthusiasm and understanding of material to which she had not been even exposed before but was now fired up to learn.

Answer 2: Several months after the start of the pilot program, a parent of a student who had always excelled in math wrote to the 6th grade pilot teacher:

> *"I just wanted to take a moment and thank you for all of your hard work this year with the 6th grade math students. Teaching is sort of like farming. You never know how your crops are going to turn out, but I have noticed with my son that his work has improved, and I really like the way he is talking about math."*
>
> *6th grade pilot parent*

The new program has worked exceptionally well on a number of students who had earlier been identified as having severe problems with learning mathematics. These students have gradually improved and excelled with the new curriculum, and questions about their future success with mathematics have become obsolete. Incidentally, a few of the girls improved so rapidly that 2 months into the program they had to be moved to the more advanced 6th-grade pilot class.

Answer 3: Here is the truth as seen through the eyes of the very first pilot teacher in the U.S. who worked with this program in 6th grade and achieved what was impossible before, and to whom the above letters were addressed:

> A pilot 6th grade teacher shares:
>
> *"This book has been a tremendous resource for my students, and I wish I had it when I was in school. The lessons are laid out in a very clear manner. The math is approachable or challenging to meet the needs of a child at any level.*
>
> *There are many math books that claim to reach every person. This book successfully does so, and it is one of the best."*
>
> *Natalya St. Clair, teacher*
>
> *co-author, "The Art of Mental Calculation"*

10. The Value of a Teacher

As with any textbooks, the present textbooks are just that: *a basis for learning.* They will come alive in the classroom *only through the teacher.*

Let us not deceive ourselves: there is no universal panacea for the middle school math problems in the U.S. And although the present textbooks have been hailed by pilot parents as:

- the best textbooks they have ever seen;
- the most suitable for their children;
- a resource to easily follow and learn from at home if a child is absent from school or did not understand something in class...

...still, *a mediocre teacher will do a mediocre job even with the best textbook in hand.*

The fact that many middle school math positions are filled by non-specialists in math is alarming and is one of the major reasons for the current general downfall of U.S. middle school math. A non-specialist does not have an "eagle view" of the full middle and high school math curriculum. Although locally, he/she might be able to teach some of the material well, a lot of damage is done by a teacher's limited knowledge and *lack of understanding of what is important in the long run, even if hard to teach.*

The situation is similar to dropping by parachute the whole math class into the middle of a jungle, giving each student a flashlight, and waiting outside the jungle to see who will come out alive. A better chance of "survival" will be ensured if the students are *given a map of the area, taught how to read and use it*, and *equipped with the necessary tools, knowledge, and skills* to efficiently cope with the difficulties that will inevitably occur on the way out of the jungle. And *the teacher has to be there, with the students, every step of the way,...* not just waiting for them outside of the "danger zone." I personally never thought of school math in this "jungle vision" until my own children experienced several different U.S. math programs and the image of the "jungle" resembled more and more powerfully the reality of pre-college math education.

More often than not, non-specialist math teachers in middle and elementary school, for a variety of reasons, decide to change the order, depth, and emphasis of the math program they are teaching from. Not knowing the "big picture" of how mathematics will develop over high school and then college years (and perhaps, not caring about this big picture, since it is not part of their day-to-day job duties), their actions unintentionally destroy the logical structure and goals of the program, whether it is a good program or not. (See earlier discussion of this situation in Section 5: The Temple of Mathematics.)

Fortunately, the new math program:

Is *straightforward to follow:* every lesson is on 2 pages only!

When each lesson in the textbook is on 2 pages only, with 6-to-10 exercises, and when the concepts are practiced in the next lessons and incorporated in new situations for better understanding, it is *very hard to justify staying on one particular lesson for a week or so* (and killing everyone's enthusiasm in the process).

(Most importantly!) **Ensures accountability from the teachers.**

The program is so crystal clear that *anyone can track where the students are and what they have covered* at any time during the school year. By "anyone" we mean parents, administrators, and random visitors alike. It is essential, though, that a copy of *the textbook resides at every student's home:* there is nothing to hide in the program, while there is so much to learn from reading the textbook at home!

Does not require any search for extra materials elsewhere: the textbook, workbook, and problem collection *contain everything for a successful completion of the program.*

Due to the *natural transparency of the material,* it is impossible to hide behind phrases like "too much," "too hard," or "not well organized" "to be covered all in one year". And the *program does depend on the steady and full completion of the material* in every middle school year, in order for the next year to be successful.

A Good Choice for a Teacher for this curriculum would take into account:

The Mathematical Level

A teacher with a *solid math background* who can *see several steps beyond middle school,* e.g., the high school curriculum, and can recognize:

- the math concepts,
- types of solutions, and
- features of analytical thinking

that are indispensable and must be taught and reinforced in middle school, despite how hard it might be to teach them to a young audience unaware of them.

The Pedagogical Level

A teacher who can *quickly adapt to new situations* and turn them into an educational advantage for the students.

A teacher who is constantly attuned to the weaknesses and strengths in students' background and skills, and who emphasizes accordingly features of the curriculum in a balanced way to match and improve the classroom situation.

A teacher who *can communicate mathematics to young students.*

The Level of Dedication

A teacher who is *dedicated to the well-being of the students* and who will sacrifice his/her own desire to "shine on the career ladder" to providing a robust math education to his/her young charges.

A teacher *who believes in and is ready to undertake and follow closely the new curriculum;* who is ready to face the mathematical and cultural difficulties that will inevitably arise in introducing such a program.

A teacher who *will persevere.*

11. Can a U.S. Middle School be Successful with this New Math Curriculum?

What kind of question is this?! Let us analyze the situation as a mathematician would, putting aside our own ambitions and de-politicizing math education.

11.1. The U.S. Gene Pool

Of course, U.S. students can do on the average just as well as (if not better than) their Bulgarian peers, and, for that matter, any peers around the world. As far as the "gene pool" of the U.S. is concerned, it is probably the best mixture in the world.

The cream of the crop of the U.S. students have performed at the top places and occasionally have made the number 1 team in the world (!) at the International Mathematical Olympiads (IMO)*, the preeminent world math competition for pre-college students that tests **not** speed but *depth and originality of mathematical thinking.*

The new math program emulates:

the IMO spirit of deep and rich math culture	mature attitude towards mathematics	dedication to long-term goals

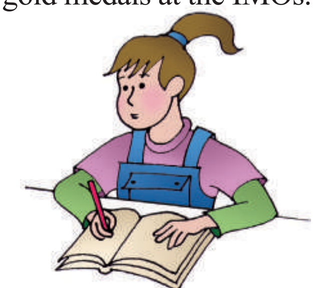
It is true that very few students will end up being math Olympians, and we will not give false hopes that the current program is training students for the IMOs. However, the expectation is that:

> Everyone who is touched by this program will build a solid Temple of Mathematics, where later, if desired, will hang the "fancy chandeliers" and "brilliant art pieces". More importantly, the personal connection with mathematics will remain all through life to be enjoyed and cherished.

11.2. Can girls succeed in math?

Upon coming to the U.S., I was shocked by this question. I was raised in a provincial town in Bulgaria and made it as a high school student to two IMOs, winning silver medals. There was another girl on the Bulgarian team for the 2 years I was there.

It took the U.S. 25 years of participating at the IMOs before the first U.S. girl, Melanie Wood, qualified for the IMOs. I was privileged to train the U.S. national team when the Melanie competed at the IMOs, also earning two silver medals. Later, I participated in the training of the other two U.S. girls, who went on to win gold medals at the IMOs.



To cut the long story short: anything is possible and *gender does not matter as far as mathematical talent and success are concerned.*

Gender apparently matters in the social and cultural aspects of education, and this severely handicaps many U.S. girls who might have otherwise become excellent mathematicians.

Decades after competing at the IMOs, I learned that Bulgaria is the top country in the world in sending the largest number of girls**, 21, to the IMOs. Germany and Russia are next, with 19 and 15 girls, respectively. USA has only 3 girls so far.

Perhaps, *it is time for the U.S. educational system to follow examples of other programs from around the world* that have been hugely successful in raising gen-

* The IMO takes place every summer in a different country. Each country's team has 6 students. As opposed to standardized tests, there are only 6 problems over 2 days, 4.5 hours each day. The solutions are written as rigorous mathematical proofs.
** See *"Cross-Cultural Analysis of Students with Exceptional Talent in Mathematical Problem Solving,"* by Titu Andreescu, Joseph A. Gallian, and Jonathan M. Kane, http://www.ams.org/notices/200810/fea-gallian.pdf .

erations of women who are highly educated in math and other STEM (Science, Technology, Engineering and Mathematics) disciplines, and who have been brought up to think of men and women as intellectually equal.

11.3. The Needed Cultural Shift

The current middle school math program originating in Bulgaria necessitates a cultural shift in a U.S. school community so that the program takes roots and yields results. For starters, the school must make:

- *math a top priority in its academic curriculum*

and follow unwaveringly in its determination to overturn the middle school math situation to the huge benefit of its students.

11.4. Continuity of the Curriculum

It is most important and advantageous for everyone that:

> The **same** teacher continues with the **same** students throughout middle school.

The alternative has struck me as a very bad way to organize middle school math education. Indeed, when a person teaches the same grade year after year he/she:

- gains no understanding of the "big picture" of pre-college math education;
- shares no responsibility for the continuation of the program in the next grade (and a lot of time will be wasted in the beginning of each school year for the new teacher to test and get attuned to the new students);
- has no incentive (or responsibility) for reaching the final goals of the program.

Yes, I am acutely aware of the counterargument:

> *"I do not want a bad math teacher to teach my kid for all of middle school. I would rather bite the bullet of a bad math teacher one year, so as to get the good math teacher the other year."*

In summary, many parents are (unwittingly) willing for their kids to endure on the average a mediocre middle school education. Can you imagine what kind of damage the "bad" teacher would do to the Temple of Mathematics during that, hopefully, single year of "bad" math lessons? Would the next, "good" teacher have the opportunity, or even the incentive, to rebuild the Temple AND cover the new material on top of that? Probably not.

This kind of "averaging math teaching" is another major reason for the tremendous difficulties in U.S. middle school math education. *A teacher must be the asset of math education:* equally important (if not more important) than the actual math curriculum! And so, the truth must be spoken:

- a bad math teacher must be let go, to prevent more irreversible damage to our kids;
- a good math teacher must be treasured and given opportunities to further develop.

11.5. Local Support

There is no easy remedy and no miracle solutions to math educational problems anywhere around the world. The new math Program will need all the support it can get from parents and students, in addition to teachers and administration. The Program will depend on the school community's belief in it and commitment to it.

Approaching the end of the first pilot year, here is what the Head of School shared, after extensive observations and conversations with pilot teachers, parents, and students:

> *"I have only heard GREAT comments about math during and after parent-teacher conferences in March. People are very pleased with the direction things are going with the Bulgarian curriculum. We are very excited to continue to grow the program next year!"*
>
> *Debra Massey*
> *Head of Pilot School*
> *Tehiyah Day School, El Cerrito, CA*

12. Within the Global U.S. Picture

At a sectional meeting of the Mathematical Association of America (MAA) in 2014, I heard a captivating presentation by *Dr. Phil Daro*, one of the three principal writers of *the Common Core Mathematics Standards* (CCSS-M). The talk addressed the biggest education problem in school mathematics, on which the CCSS-M writing team focused:

> *"The mile wide, inch deep"* configuration of the American school curriculum.

As a consequence, Dr. Daro would ask any teacher:

"Are you teaching

Mathematics or *Answer-Getting Strategies?"*

This question fully resonated with the educational philosophy with which I was brought up back in Bulgaria. Dr. Daro explained that the core change incorporated in the CCSS-M is:

> *A shift in the perspective* toward mathematics taught in schools:

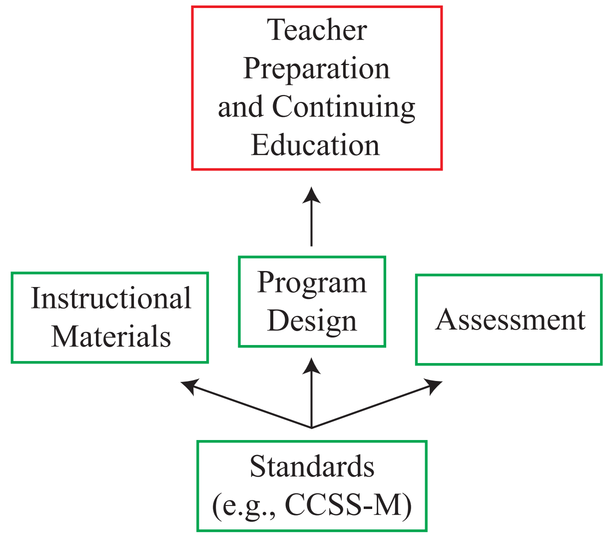
Less emphasis on a fragmented multitude of special solution methods, especially tricks that avoid underlying mathematical principles.

More emphasis on core principles and practices that unify and focus mathematical thinking.

Dr. Daro assembled the major pieces of the gigantic math education picture in the U.S.:

> *"Standards can help solve this kind of curriculum architecture problem, but we all face more detailed and difficult challenges of instructional materials, assessment, and program design, as well as the even more [difficult] area of teacher preparation and continuing education."*
>
> *Dr. Phil Daro*
> *A Principal Writer*
> *Common Core Mathematics Standards*

Visually, the scheme below represents the levels of change in the U.S. math education that are envisioned by the CCSS-M writing team:



At the time, I had no idea that I would soon launch the translation, adaptation, and implementation effort of the present curriculum. Inspired by Dr. Daro's talk, I invited him to meet in person the then-leaders of Tehiyah Day School. Perhaps, not entirely by serendipity, this meeting marked the path for the new math curriculum: as it turned out, the new Bulgarian curriculum matched very closely the CCSS-M.

During the 2016-2017 academic school year, I spent a day every week working with the pilot classes, training the pilot teachers, and learning myself how to adapt the curriculum to the U.S. reality. The instructional materials, program design, and assessment of the new adapted math curriculum for 6th grade are ready, and the adaptations of the 7th and 8th curricula are in preparation.

Coming full circle, two years after his memorable MAA presentation Dr. Daro reviewed the U.S.-adapted version of our new math program and assessed it as a

> *"Middle and High School Math Program that is deeply aligned with the Common Core Mathematics Standards".*

13. Math 2 (7B) Program by Topics and Lessons

Algebra

Linear Equations

- Numerical Equalities L94
- Equations with One Unknown L94
- Equivalent Equations L95
- Identities L97
- Types of Equations:
  - $ax + b = 0$ L96
  - $(ax + b)(cx + d) = 0$ L98
  - $|ax + b| = 0$ L99
- Reducing to Linear Equations L100
- Linear Parametric Equations L101
- Linear Equations (Review) L151
- Proportions (Review) L147
- Polynomials (Review) L148

Geometry

Foundations of Geometry

Basic Geometric Concepts:

- Point, Line, Segment L79
- Ray, Half-Plane, Angle L80

Angles:

- Supplementary and Vertical Angles L81-82
- Exterior Angle of a Triangle L89

Lines:

- Perpendicular Lines L81-82
- Transversal Intersecting Two Lines L83
- Parallel Lines. Axiom. Criteria L83-86*

Triangles:

- Concept. Working w/ Triangles L87-90
- Sum of Angles L88
- Geo-Foundations (Review) L154
- Solids (Review) L145-146

Congruent Triangles

- Definition L114

Criteria for congruence:

- SAS L115-118
- ASA L117-118
- SSS L123
- SSR L129

Special Triangles:

- Isosceles L119-120,132
- Equilateral L120
- 30°- 60°- 90° L125-126

Perpendiculars:

- Perpendicular Bisector L121-122
- Perpendicular from a Point to a Line L124

Special Segments in Triangles:

- Median to the Hypotenuse L127-128
- Angle Bisector L130-132
- Altitude, Median, and Angle Bisector in an Isosceles Triangle L132
- Congruent Triangles (Review) L155

Euclidean Constructions

- Geometric Constructions L136
- w/ Ruler & Compasses L137-138
- Triangles by SAS L139
- Triangles by ASA L140
- Constructions (Review) L155

Geo-Algebra

Algebra in Geometry:

- Review of Surface Area and Volume of Solids L145-146,149-150
- Area/Perimeter Calculations L87,117,121
- Angle Calculations L88-93,119-122,125-128,131-135

Geometry in Algebra Word Problems:

- Diagrams on Motion L106-107,112
- Histograms/Circular Diagrams in Everyday Problems L152-153,158-159
- Graph Preview L160

Applications

Modeling with Linear Equations

- Introduction to Modeling L104-105
- Modeling (Review) L151-153

Word Problems:

- On Motion L106-107
- On Work L108-109
- On Interest Rates and Capital L110
- On Mixtures & Alloys L111
- From Everyday Life L151-152
- Obstructed Distance L134

Averages:

- Arithmetic, Geometric, and Harmonic Means L160

Inequalities:

- Arithmetic Mean - Geometric Mean Inequality L160
- Geometric Mean - Harmonic Mean Inequality L160

Legend

- Algebra
- Geometry
- Applications
- Geo-Algebra
- Logic and Proofs

* L83-86 = Lessons 83 through 86

Logic

Basic Logic Structure of Mathematics:

- Sub-areas of Geometry L78
- Concepts, Definitions, Primitive Concepts L78
- Statements. Primitive Statements, Axioms L78
- Theorems as Statements L78
- Proof of a Theorem L78, 81
- The Axiomatic Method L78
- Proof Problems L78,114,117-118,120-122,124,128,132,139-140
- Solution to a Problem, Correct Solution L78
- Hypothesis - Conclusion; If - Then; Given - Prove L81,114,133
- Proof by Contradiction L85
- Proof by Extra Construction L90-92
- Equivalent Statements, If and Only If L133
- Problems of Impossibility L143

Build a Logic Network in Geometry:

- Primitive Concepts in Geometry L78-79
- Axioms in Geometry L78,81,85
- Basis Geometry Concepts L79-80,86-87
- Basic Definitions of Angle Pairs L81-83,89
- Basic Theorems in Geometry L81-89
- Basic Definitions for Pairs of Lines L82-83,124
- Basic Proof Problems in Geometry L82
- Segments in a Δ L87,90,121-122,127,130,132
- Basic Terminology/Theorems of Congruent Δ's L114-115,117,119-125,127,129-130,132-133
- Centers in a Δ L122,130,141,143-144
- Special Perpendicular Lines L121-124
- Basic Tools and Constructions L136
- Basic Construction Problems L137-141

Build a Logic Network in Equations:

- Numerical Equalities: True/False, Equivalent L94
- Equations: Unknown, Constant Term, Root, First Degree L94-95
- Theorems for Equivalent Equations L95

Substituting Parameters into Equations L101

Recognize and Distinguish:

- Supplement vs. Supplementary L81
- Implicit vs. Explicit Hypothesis & Conclusion L82
- Criterion vs. Property L83-84,114-120,123,129
- Coincide vs. Intersect vs. Parallel (Lines) L83
- Alternate vs. Corresponding vs. Consecutive Pairs of Angles L83
- Interior vs. Exterior:
  - Angle Pairs L83
  - Point of a Figure L87
  - Angle of a Triangle L89
- Axiom A_4 vs. Euclid's 5^{th} Postulate L85
- Exists vs. Unique L85
- Corollary vs. Theorem L85-88
- Direct vs. Indirect Proof L85
- Congruent vs. Equal L114
- Analogous Statements, Proofs, Solutions L82,88-89,116,118,124,128,132,134
- Arbitrarily Chosen vs. Special Object L121
- Converse Statements L121,125,127,130
- Construction Algorithm vs. Justification L137-141
- Left-Hand Side vs. Right-Hand Side L94
- Canceling vs. Moving Terms in Equations L95-96
- Equations vs. Equalities vs. Identities L97

Problem Solving

Search for Multiple Solutions in:

- Geometry and Constructions L83-84,91,122
- Equations and Modeling L98, L105-108

Split into Cases:

- Geometry L85,116,119,121,123,129,138,144
- Algebra L101-102,104

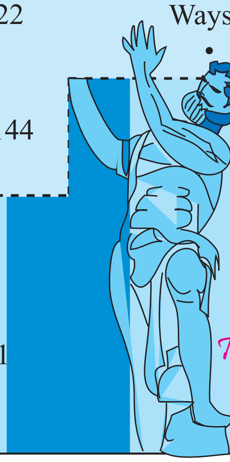
Read, Interpret, and Solve Word Problems With:

- Making a Table L104-111
- Diagram L106-107,109,112
- Express the Same Quantity in Two Different Ways and Equate L96,104-112
  - Geometric Pictures L78-93,114-135

Creating and Writing Proofs

Recognize Basic Logic Structures:

- Axiomatic Method L78
- Equalities, Equations, Identities L94-97,101
- Problem-Solving Ideas L112
- Congruences are Everywhere L114



Study and Review

Review: 7A Material L145-148
7B Material L151-155
Summarize: L91,102,112,134,141
Practice: L92,143-144,152-155
Chapter Tests: L93,103,113,135,144-148
Tests w/ Solutions: L90,122,149,156
General Tests: L149-150
Exit Level Tests: L157-159

14. How to Work w/ Math 2 (7B) Materials

The Math 2/7B Textbook is for use in class as the main source of material to be covered:

- *The Textbook Examples* are the core of the class discussion and work.
- *The Textbook Exercises* can be done in class or assigned for homework.
- *Answers to all Exercises* are in the back of the textbook; hence, writing only answers will bring no credit, while good reasoning and organized calculations will earn a lot of credit.

All Lesson Exercises in the Math 2/7B Workbook (marked as L__) are identical to the examples in the textbook, with spaces for writing solutions. The Lesson Exercises are intended for classwork.

The Worksheets in the Math 2/7B Workbook (marked as W__) thematically match the textbook, lesson-by-lesson. The Worksheets are intended mostly for independent work in class or homework. They are usually (but not always!) geared towards students who need more help and extra support with the material.

The Math 2/7B Problem Collection for the Teacher can be used for tests and supplementary work with students who excel with the material.

The Math 2/7B Pedagogical Guide will reveal to the teacher ideas and strategies gained by decades of teaching in the U.S. and by the practical wisdom and experience of the Program's pilot teachers.

15. Biographical Sketches

The idea for the *Archimedes Publishing House* was conceived in 1982 when Professor Georgi Paskalev collected all entrance exams given thus far to Mathematics Departments in universities across Bulgaria and wrote a book with their solutions. The occasion was the upcoming exam of his own son, who actively participated in the preparation of the book. Until 2004, Professor Georgi Paskalev continued to study the competitive university entrance exams and published a variety of preparation materials. These books have enjoyed continual interest, large circulations, and many editions.

In 1989 Professor Georgi Paskalev, together with his wife Zdravka Paskaleva and grown-up children Plamen and Nadezhda,founded a Publishing House *"Georgi Paskalev,"* specializing entirely in mathematics materials. Ten years later Plamen Paskalev assembled a team of authors and published a Collection of Tests to prepare for admissions to high schools. This book became the first publication of *"Archimedes,"* the successor to "Georgi Paskalev".

The new Publishing House is directed by Plamen Paskalev, and it still bases its activities on the creative tandem of Georgi Paskalev and Zdravka Paskaleva. During 1997-2004, these two authors created a complete series of math textbooks for 5^{th}-12^{th} grades, which set new standards in the methodology of teaching mathematics and has ever since enjoyed popularity in teaching circles around Bulgaria.

Over the years, Archimedes Publishing House grew to include books in informatics, information technology, and chemistry, as well as translated academic and journalistic literature. "Archimedes" is an active member of the *Association Bulgarian Book* and an active participant at the publishing conferences of the *Union of Mathematicians in Bulgaria.*

In 2010 the *European Council for Scientific and Cultural Community* awarded the *"Golden Book"* to "Archimedes" for its contributions to the development of Bulgarian culture.

Professor Georgi Paskalev (1932-2004) taught for many years at Sofia University. He co-founded the Mathematics Department at Plovdiv University (PU) and was the Vice Provost of PU (1970-1972), and a longtime Chair of the Departments of Mathematics and Mechanics at PU and the University of Structural Engineering & Architecture (USEA) in Sofia.

Professor Paskalev was known for his ability to teach difficult math disciplines in a clear and accessible manner. He published numerous research papers in various areas of mathematics and mechanics. He mentored many mathematicians and authored more than 100 books, problems collections, and textbooks in mathematics. In 2004 the Academic Council of PU named its auditorium after him. Another auditorium at USEA in Sofia is also named in his honor.

Zdravka Paskaleva (1938-2017) was a well-known and experienced specialist in mathematics and the methodology of mathematics, with experience of 20 years as teacher in elite high schools and 10 years as a leader in teachers' education and development, having observed and reviewed over 1,000 lessons by teachers at various schools. From 1985 until 2010 she was the head of the *Education Center "Archimedes"*.

Maya Alashka received a degree in mathematics from Sofia University (SU), with a specialty in elementary school education. She has taught math for 40 years at all pre-college levels. As an affiliate of SU, she has trained many future teachers. She has participated in various education projects and co-authored a number of popular textbooks, problem collections, and other education materials for all pre-college levels.

Plamen Paskalev is an Assistant Professor of Mathematics, author of dozens of scientific publications in mathematics and mechanics, collections and guides for students, and articles on the challenges in math education. He has taught at USEA, the University of National and World Economy, and the German Language High School in Sofia. He has directed "Archimedes" since its creation in 1999. In 2004 he founded and has since chaired the *Union of Publishers of Textbooks & Curriculum Materials in Bulgaria*.

Professor Zvezdelina Stankova (Zvezda) was drawn into the world of mathematics when, as a 5th grader, she joined the math circle at her school in Bulgaria and won, three months later, the Regional Math Olympiad. She represented her home country at two *International Mathematical Olympiads* (IMOs), earning silver medals.

As a freshwoman at Sofia University, Zvezda won a competition to study in the U.S. and completed her undergraduate degree at Bryn Mawr College in 1993. She did her first math research in enumerative combinatorics at two summer programs in Duluth, Minnesota. The resulting papers contributed to her *Alice T. Schafer Prize* for Excellence in Mathematics by an Undergraduate Woman, awarded by the *Association for Women in Mathematics*. In 1997, Zvezda received a Ph.D. from Harvard University, with a thesis on moduli spaces of curves, in the field of algebraic geometry. She also earned a high school teaching certificate in the state of Massachusetts and later in California.

As a postdoctoral fellow at the Mathematical Sciences Research Institute (MSRI) and UC Berkeley in 1997-1999, Zvezda co-founded the *Bay Area Mathematical Olympiad* and created the *Berkeley Math Circle* (BMC). She trained the USA national team for the IMOs for six years, including the memorable year 2001 when three of the six team members were BMCers, and USA tied with Russia for a second overall place in the world. She worked at Mills College 1999-2016, and she joined the faculty at UC Berkeley in the summer of 2016.

Zvezda's inspiring style and passion to teach were recognized by the *Mathematical Association of America* (MAA): in 2004 she received the first *Henry L. Alder Award* for Distinguished Teaching by a Beginning College or University Mathematics Faculty Member. In 2011 MAA honored her with the highest math teaching award in the U.S., the *Deborah and Franklin Tepper Haimo Award* for Distinguished College or University Teachers who are widely recognized as extraordinarily successful and whose teaching effectiveness has shown to have influence beyond their own institutions.

Zvezda was featured in the *Salutes Program of the ABC 7 News* in spring 2011. In 2012, she was listed in *Princeton's Review "300 Best Professors."*

In fall 2015 she introduced a new middle school math program at Tehiyah Day School in El Cerrito, CA, and in fall 2016 she founded "Math Taught the Right Way" program for middle and high school students at UC Berkeley. Both programs are based on this textbook series.

Zvezda's most enduring passion remains working at the BMC with young students motivated to discover new mathematical wonders. She spends a lot of time with her girl and boy, studying with them foreign languages and playing the piano, and teaching them mathematics the "Bulgarian" way.

Tom Rike graduated from San Francisco State College in 1968. He taught in the Oakland Unified School District from 1970 until he retired in 2003 and now volunteers at Oakland High School three days a week. He served as the high school liaison for BMC and BAMO until 2009.

Tom has been fascinated by giants of mathematical thought such as *Archimedes*, *Euler*, and *Gauss*, and has shared their works at the BMC sessions on a number of occasions: using the *arbelos* from the *Book of Lemmas* by Archimedes; showing Euler's solution to the *Basel Problem*; and demonstrating Gauss's proof from *Disquisitiones Arithmeticae* that a 17-gon is *constructible*. It is Archimedes' *lever*, on which he "balanced" his *Mass Point* article in Volume I of *"A Decade of the Berkeley Math Circle - the American Experience"*. For a number of years, Tom also ran a math circle at Oakland High.

Tom has been involved as a coach and director of the *East Bay Mathletes*, a monthly competition among local high schools since 1978. In 1983, he helped found in Oakland a middle school math competition, *All-Star Mathletes*.

Stankova's and Rike's bios of are partially taken from "A Decade of The Berkeley Math Circle," AMS/ MSRI's Math Circles Library.

Among his other interests are the game of go, opera, chamber music, languages, and the writings of *James Joyce*.

The current series of textbooks has tremendously benefited from Tom's unsurpassed expertise in U.S. pre-college mathematics and from his generous and unbounded enthusiasm for new and sound ways to teach mathematics to youngsters.

Kelli Talaska has been a student and teacher of mathematics nearly all her life, starting with frequently volunteering to write the problem of the day in fifth grade. Her mathematics education really took off in her years as a student and later a counselor at the *Honors Summer Math Camp* (for high school students) at Texas State University. She spent two summers working on original mathematics research at the *Duluth Summer Program* for undergraduates. After completing her degree at Texas A&M University in 2002, Kelli spent two years teaching at Lowell High School in San Francisco.

She later returned to graduate school at the University of Michigan and received her Ph.D. in 2010, with a thesis in algebraic combinatorics. Kelli was an NSF Postdoctoral Fellow at UC Berkeley from 2010-2013, and returned to UC Berkeley as a lecturer in 2015, after starting her tutoring and enrichment business, *Octopus Math*.

Kelli relishes the chance to work with students of all ages and to expose students to the idea that mathematics involves not just algorithms and memorization, but also incredibly rich opportunities to experiment and to look for patterns and connections. She has been involved with the *Berkeley Math Circle* since 2015 and has also had the opportunity to teach at the *Santa Cruz Math Circle* and at *Epsilon Camp*.

Kelli really enjoys hands-on math explorations, including logic puzzles, geometric constructions, and paper cutting and folding activities. Outside of math, Kelli is an avid scuba diver and loves to talk about fascinating ocean critters.

16. Math Taught the Right Way

In addition to running the Tehiyah Day School math program on this textbook series for a third-year, we continued for a 2nd year a weekly program *"Math Taught the Right Way"* (MTRW), which extended the Berkeley Math Circle at UC Berkeley.

MTRW in fall 2017 and spring 2018 had an increased enrollment of 100 students from 62 schools and 17 towns, mostly in the eastern part of the San Francisco Bay Area. The three program groups were selected mainly by age. As in the spirit of the Berkeley Math Circle, younger, more advanced students were promoted to higher groups. The fall groups were named 6A, 7A, and 8A, and their spring successors 6B, 7B, and 8B.

6A/6B Group:	**Taught by:**
• 6A Problem Solving (5-6pm)	Ellen Kulinsky
• 6B Problem Solving (5-6pm)	Elena Blanter
• 6A/6B Geometry (6-7pm)	Hai Main-Luu
• 6A/6B Algebra (7-8pm)	Elysee W.-Egolf
7A/7B Group:	
• 7A/7B Problem Solving (5-6pm)	Elysee W.-Egolf
• 7A/7B Geometry (6-7pm)	Norm Prokup
• 7A/7B Algebra (7-8pm)	Kelli Talaska
8A/8B Group:	
• 8A/8B Proofs (5-6pm)	Zvezdelina Stankova
• 8A/8B Geometry (6-7pm)	Norm Prokup
• 8A/8B Algebra (7-8pm)	Kelli Talaska

The 14 classes in pink were taught exclusively from the current adapted curriculum. In the 15 weekly 3-hour sessions during the fall of 2017,

- 6A group covered all of the 6A curriculum. 6A Problem Solving concentrated on the *Divisibility* chapter and *Word Problems* from the 6A curriculum.
- 7A Algebra covered *Proportions* and *Integer Expressions*. 7A Geometry worked on *Polyhedral* and *Round Solids* in the 7A curriculum.
- 8A Algebra covered *Functions* and returned to review *Square Roots* and *Quadratic Equations*. 8A Geometry covered half of the *Circle Geometry* chapter in the 8C curriculum.

In the 15 weekly 3-hour sessions in spring 2018,

- 6B group will complete the whole 6B curriculum. 6B Problem Solving will concentrate on *Applications of Fractions* and *Word Problems*.
- 7B Algebra class will cover *Algebra of Polynomials* and half of *Equations* in the 7A-7B curriculum. 7B Geometry will complete *Foundations of Geometry* and half of *Congruent Triangles*.
- 8B Algebra class will work on *Systems of Equations and Inequalities*, while 8B Geometry will finish *Circle Geometry* and cover *Isometries* from the 8A curriculum.

The four classes in blue were done in a style resembling math circle sessions.

- 7A/7B Problem Solving work its way through *Bulgarian Math Contests* for 7th grade, picking up topics, ideas, and theory related to each problem and learning all necessary material.
- 8A/8B Proofs are based on *"A Decade of the Berkeley Math Circle - the American Experience,"* edited by me and Tom Rike, and expose students to the art of proofs within the context of math circle material: *Number Theory, Part I* (vol. I) and *Multiplicative Functions, Part I* (vol. II).

The Berkeley Math Circle Summer Program 2017 could be partially viewed as a third component of the MTRW Program, which completes the 6th-8th grade curriculum, plus some math circle topics. The faculty who taught at the program are:

- Dmitry Fuchs, Ph.D., UC Davis
- Elysee W.-Egolf, B.A. UCB, Bentley High
- Evan O'Dorney, BMC alumnus, Princeton
- Hai Main-Luu, BMC asst, UC Berkeley
- Kelli Talaska, Ph.D., UC Berkeley
- Laura Pierson, BMC alumna, Harvard
- Norm Prokup, Ph.D., College Prep, Oakland
- Petra Munih, Ph.D. Biophysics/Structural Biology
- Pratima Kapre, Ph.D., Santa Clara University
- Quan Lam, Ph.D., MBA, Office of UC President
- Sam Vandervelde, Ph.D., Proof School
- Zvezdelina Stankova, Ph.D., UC Berkeley

17. Acknowledgments

The adaptation to the U.S. reality of the language and math notation in this middle and high school math program, as well as the technical and mathematical correctness of the series, is due to the dedication and selflessness of *Tom Rike,* my long-time collaborator and co-editor, and *Kelli Talaska,* an instructor at BMC and MTRW. You are both *amazing* (and that's not hyperbole)!

Natalya St. Clair was the very first pilot teacher, who moved from across the country to bravely undertake the new math Program in fall 2015 with the two 6^{th} grade classes at Tehiyah Day School. Her enthusiasm for mathematics was evident each and every day in her lively lessons, sharing with her students her joy in teaching math. She also had numerous suggestions for adaptation, based on her daily experience with the Program on the U.S. turf, and offered constructive criticism of the revised version of the series. *George Khasin* and *Laurie Gold* continued the pilot Program in spring 2016 with a steady rhythm and care for the students, and *Andrea Jedwab* piloted the 7^{th} and 8^{th} grade and continued with the 6^{th} grades classes in 2016-2018.

Going further back, *Elise Prowse,* the former Interim Head of the pilot school and a math educator herself, recognized the power and depth of the new Program (while it was still in the original Bulgarian language!) and laid down the path of its implementation in practice. *Debra Massey,* the former Head of School and a parent herself, strongly backed the Program and supported it all the way through the crucial first two pilot years.

The middle school math Program at Tehiyah Day School would not have had a stable basis to land on, had it not been for the Singapore Math Program in grades K-5, introduced in 2003 by *Professor Alexander Givental* of UC Berkeley and his wife, *Laura Givental,* the leader of the Berkeley Math Circle-Elementary Division for grades 1-4. The school can and will improve the efficiency of implementing the Singapore Math Curriculum at the elementary school grade levels. Still, the mere fact that Singapore Math precedes the pilot Math Program already immensely increased the chances of success of this pioneering Project.

It goes without saying that the vision of *Professor Georgi Paskalev* and his collaborators, *Zdravka Paskaleva* and *Maya Alashka*, has provided the generating power for this translation and adaptation effort. The success of the original Bulgarian textbook series is no surprise. The novel ideas of overall design, the simplicity and elegance in organization, and the balance between depth and breath of the material, are unsurpassed.

Any textbook series needs a Publisher. And what more suitable Publisher of the new textbook series is there than the original "Archimedes" Publishing House! Its Director, *Professor Plamen Paskalev,* undertook the Project without reservations, investing in a future based on the common ground of two very different math educational systems: the U.S. and the Bulgarian systems.

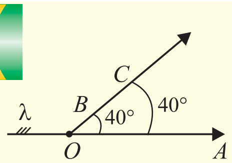
Angelina Avramova, the graphic designer of the original series and the technical editor of the new series, prepared the materials for publication with efficiency and professionalism that would put any anxious collaborator at ease and that would make any publisher proud.

Looking forward, *Dr. Phil Daro,* one of the three Principal Writers of the Common Core Standards, and *Dr. Oleg Gleizer,* the Academic Dean of Geffen Academy at UCLA and Chief Curriculum Developer and Lead Instructor at the Los Angeles Math Circle, reviewed and endorsed the adapted Program. *Dr. Oleg Gleizer* and *Dr. Sibyll Catalan,* the Head of Geffen Academy, are working towards adopting the Program there. *Dr. Doug Ensley,* the Deputy Executive Director of the American Math Competitions made available about a dozen AMC8 problems for inclusion in the current 7B Textbook.

The daily work of the pilot *6^{th}* and *7^{th} grade students* and *the constant support of their parents* at Tehiyah moved this program from paper to real life and gave it meaning and purpose. The translation and adaptation effort would have been inconceivable without the understanding and love of my family: my husband, *Dima,* and our children, *Lily* and *Oleg.*

With gratitude,
Zvezdelina Stankova
Translator, Adapter, and Co-Author

81. Supplementary Angles. Right Angles



If in half-plane λ with contour AO we draw with protractor $\angle AOB = 40°$ and $\angle AOC = 40°$, we will see that the rays $\overrightarrow{OB}$ and $\overrightarrow{OC}$ coincide.

A3 **Axiom for unique transport of an angle:** If two equal angles with a common side are laid in the same half-plane with contour this common side, then their second sides will also coincide.

Supplementary Angles

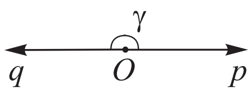


Fig. 24

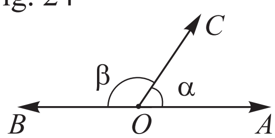


Fig. 25

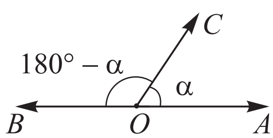


Fig. 26

Recall that a **straight angle** is an angle whose sides are opposite rays. In Fig. 24 $\angle(p, q) = \gamma = 180°$ is a *straight* angle.

Moving on:

D21 **Linear pairs.** Two angles with one common side and other sides that are opposite rays are called a **linear pair**.

From the definition of a sum of two angles it follows that $\angle AOC + \angle COB = \angle AOB$, and hence $\alpha + \beta = 180°$ (Fig. 25). β is called the **supplement** of α and vice versa.

D22 **Supplementary angles.**
Two angles that add up to 180° are called **supplementary angles**.

The above calculations justify the following theorem:

T1 Angles in a linear pair are supplementary.

For supplementary angles α and β, the following equalities are true: $\alpha + \beta = 180°$, $\alpha = 180° - \beta$, and $\beta = 180° - \alpha$ (Fig. 25-26).

Example 1 Angles α and β are supplementary. Find angle β if:
a) $\alpha = 30°$; b) $\alpha = 47°30'$; c) $\alpha = 32°18'10''$.

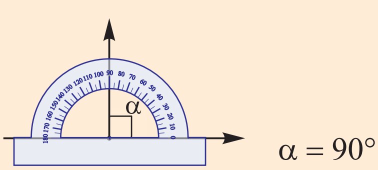
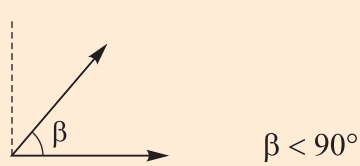
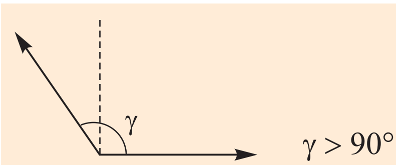
Solution:

a) $\beta = 180° - \alpha =$
$= 180° - 30° =$
$= 150°$

b) $\beta = 180° - \alpha =$
$= 180° - 47°30' =$
$= 179°60' - 47°30' =$
$= 132°30'$

c) $\beta = 180° - \alpha =$
$$\begin{array}{rrrr} = & 179° & 59' & 60'' \\ - & 32° & 18' & 10'' \\ \hline & 147° & 41' & 50'' \end{array}$$

Types of Angles

Right angle	Acute angle	Obtuse angle
 An angle with measure 90° is called a **right angle**.	 An angle with measure smaller than 90° is called an **acute angle**.	 An angle with measure between 90° and 180° is called an **obtuse angle**.

It is customary to denote a right angle with a square aligned at the vertex of the angle, as shown in the first drawing above. To draw a right angle we can use a right triangle tool.

Example 2 What is the degree measure of an angle that is four times as large as its supplement?

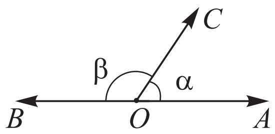


Fig. 26

Solution: Let β = original angle and α = β's supplement.

$\beta = 4\alpha$ (by hypothesis);

$\alpha + \beta = 180°$ (supplementary angles);

$\alpha + 4\alpha = 180°$, $5\alpha = 180°$, $\alpha = 36°$, and $\beta = 4 \cdot 36° = 144°$.

Properties of Supplementary Angles

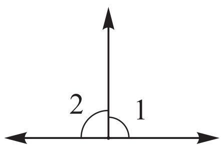
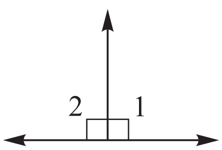


Fig. 27

T2 If one of two supplementary angles is right, then the other is also right.

Given: $\angle 1$ and $\angle 2$ are supplementary angles (Fig. 27), $\angle 1 = 90°$.
Prove: $\angle 2 = 90°$.
Proof: $\angle 1 = 90°$ and $\angle 2$ (supplementary) $\Rightarrow \angle 2 = 180° - 90° = 90°$.

T3 If two supplementary angles are equal then both of them are right.



Proof (Fig. 27):
Angles $\angle 1$ and $\angle 2$ are equal. Thus, they have the same measure α.
Angles $\angle 1$ and $\angle 2$ are supplementary.
Then $\angle 1 + \angle 2 = 180° \Rightarrow \alpha + \alpha = 180°, \alpha = 90°$
$\Rightarrow \angle 1 = \angle 2 = 90° \Rightarrow \angle 1$ and $\angle 2$ are right angles.

Remember!

The formulation of a theorem has two parts: the **hypothesis** is what follows "if" and the **conclusion** is what follows "then." When writing the proof we make a drawing and introduce notation; and using the notation we write down the hypothesis (**"Given"**), the conclusion (**"Prove"**), and the **reasoning of the "Proof."** In the course of the proof, we use concepts and theorems that have been already introduced/proved.

In this lesson

We introduced:

- supplementary angles;
- right, acute, and obtuse angles.

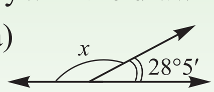
T1: Angles in a linear pair are supplementary.
T2: The supplement of a right angle is also a right angle.
T3: Two equal supplementary angles are both right.

Exercises

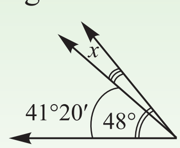
1 If $\angle AOB = 135°27'$ and $\overrightarrow{OL}$ is its angle bisector, find the measures of the two angles $\angle AOL$ and $\angle LOB$.

2 Find the measures of the angles denoted by x in the drawings:

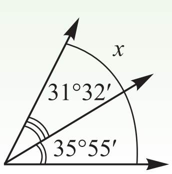
a)



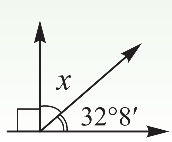
b)



c)



d)



3 How many degrees is the supplement of angle α if:
a) $\alpha = 75°$; b) $\alpha = 108°$;
c) $\alpha = 135°$; d) $\alpha = 90°$?

4 How many degrees is the supplement of angle α if:
a) $\alpha = 38°20'$; b) $\alpha = 125°55'$;
c) $\alpha = 70°36'45''$; d) $\alpha = 137°58'59''$?

5 How many degrees is the angle that is:
a) 30° larger than its supplement;
b) $^1/_3$ of its supplement;
c) 20% of its supplement?

96. The Equation $ax + b = 0$. Part I

The Equation $ax + b = 0$ when $a \neq 0$

Example 1 Solve the equations:

a) $2x+10=0$; b) $9x+3=2x+5$; c) $4.3x+2=2x+1\div 0.5$.

Solution:

a)
$$2x+10=0$$
$$2x=-10$$
$$x=-5$$

b)
$$9x+3=2x+5$$
$$9x-2x=5-3$$
$$7x=2$$
$$x=\frac{2}{7}$$

c)
$$4.3x+2=2x+1\div 0.5$$
$$4.3x-2x=2-2$$
$$2.3x=0$$
$$x=0$$

$\mathbf{2x=-10}$	$\mathbf{7x=2}$	$\mathbf{2.3x=0}$
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$
$\mathbf{2x+10=0}$	$\mathbf{7x+(-2)=0}$	$\mathbf{2.3x+0=0}$

The equations that we solved in *Example 1* have one unknown, x, and can all be written in the form $\boldsymbol{ax+b=0}$.

D7 An equation of the form $\boldsymbol{ax+b=0}$ where x is the unknown number, and $a \neq 0$ and b are numbers, is called **an equation of first degree with one unknown.**

Any equation of first degree with one unknown always has a solution, and this solution is unique.

$$\boldsymbol{ax+b=0 \iff ax=-b \iff x=-\frac{b}{a}} \quad \textbf{when } \boldsymbol{a \neq 0}$$

Example 2 Solve the equation $6x^2-(2x-1)(3x+1)=3(1-x)$.

Solution:

$$6x^2-(2x-1)(3x+1)=3(1-x)$$
$$6x^2-(6x^2+2x-3x-1)=3-3x$$
$$\cancel{6x^2}-\cancel{6x^2}\underbrace{-2x+3x}+1=3-3x$$
$$x+1=3-3x$$
$$x+3x=3-1$$
$$4x=2 \quad |\div 4$$
$$x=\frac{1}{2}$$

1. We simplify the expressions on both sides of the equation and rewrite them as equivalent polynomials.
2. We move the terms that contain x over to the LHS and the constant terms over to the RHS.
3. We simplify and obtain an equation of the form $ax+b=0 \Leftrightarrow ax=-b$ where $b=-2$.
4. We divide both sides of the equation by the coefficient in front of x.

Example 3 Solve the equation $(2x+1)^2-3=4(x^2+1)+x$.

Solution:

$$(2x+1)^2-3=4(x^2+1)+x$$
$$4x^2+4x+1-3=4x^2+4+x$$
$$\cancel{4x^2}+4x-\cancel{4x^2}-x=4-1+3$$
$$3x=6 \quad |\div 3$$
$$x=2$$

If there are equal terms in the two sides of the equation, after moving one of them over to the other side these terms become opposite and cancel each other. We can also cancel them quickly beforehand, while they are still on different sides of the equation:

$$\cancel{4x^2}+4x-2=\cancel{4x^2}+4+x.$$

The Equation $ax+b=0$ when $a=0$

Example 4 Find a number with the property:

"The difference of the square of the number and double the number is equal to the square of the difference of this number with the number 1."

Solution:

We denote the desired number by x and obtain

$$x^2-2x=(x-1)^2$$
$$x^2-2x=x^2-2x+1$$
$$\mathbf{-2x+2x=1}. \qquad (1)$$

In equality (1) the terms that contain x cancel each other; i.e., this equality does not contain x. The question now arises whether this equality is an equation and whether it is true:

$-2x+2x=1 \Leftrightarrow (-2+2)x=1 \Leftrightarrow 0\cdot x=1$; i.e., $ax=b$ when $a=0$?

To solve problems similar to *Example 4*, an equality of the form $ax+b=0$ is thought of as an equation even when $a=0$.

- **To solve the equation**
  $0\cdot x=5$,
  we look for x so that $0\cdot x=5$.
  But there is **no** such number!
  $\Rightarrow$ The equation $0\cdot x=5$ has **no** solutions.

- **To solve the equation**
  $0\cdot x=0$,
  we look for x so that $0\cdot x=0$.
  But **any** number multiplied by 0 is 0.
  $\Rightarrow$ **Any** number is a root of the equation $0\cdot x=0$.

In *Example 4*: $-2x+2x=1 \Leftrightarrow (-2+2)x=1 \Leftrightarrow 0\cdot x=1 \Rightarrow$ The equation has no solutions.
Thus, the problem has no solutions either; i.e., there is **no** number with the desired property.

The equation $0\cdot x=b$ | **when $b\neq 0$ is $0\cdot x=b$ and it has no solutions;**
when $b=0$ is $0\cdot x=0$ and every number is a root.

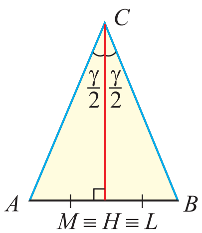
D8 Equations of the form $ax+b=0$ are called **linear equations.**

Exercises

Solve the equations:

1. $5(1-x)=-10x$;
2. $2(x+4)=4(x-1)$;
3. $2(x+3)+3x=4x-5$;
4. $3(x-2)-4(x+3)=2x+3$;
5. $2(3-x)+(x-1)-15=3(2x-1)$;
6. $4(x-2)-7=4x+3(1-2x)$;
7. $3-[x+1-(x-6)]=4x+5$;
8. $5-x-[-x+2(x-1)]=4x+5$;
9. $(x+4)^2-3(x+2)=x(x+2)$;
10. $(2x-1)^2-3x(x+3)=(x+1)^2$;
11. $(x+2)^2-(x-3)(x+3)=2x-7$;
12. $(x+2)^3-x(x+1)^2=(2x+3)^2$.

132. Altitude, Angle Bisector, and Median in an Isosceles Triangle



We know that in an isosceles ΔABC ($CA = CB$):

- the median, the altitude, and the angle bisector to the base AB coincide and lie on the perpendicular bisector of AB.

Example 1 Given is an isosceles $\Delta\, ABC$ ($AC = BC$). Prove that:

a) If AH_1 and BH_2 are altitudes, then $\Delta ABH_1 \cong \Delta BAH_2$ and $AH_1 = BH_2$.
b) If AM_1 and BM_2 are medians, then $\Delta ABM_1 \cong \Delta BAM_2$ and $AM_1 = BM_2$.
c) If AL_1 and BL_2 are angle bisectors, then $\Delta ABL_1 \cong \Delta BAL_2$ and $AL_1 = BL_2$.

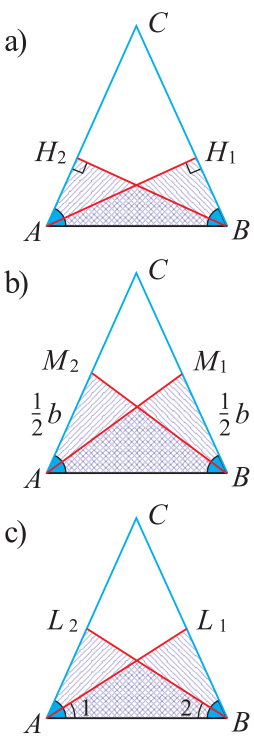


Fig. 84

Solution:

Given is ΔABC with $AC = BC$, so $\angle A = \angle B$.

a) (Fig. 84a) We consider ΔABH_1 and ΔBAH_2 | AB common side; $\angle CAB = \angle CBA$ (given); $\angle H_1 = \angle H_2 = 90°$

$\Rightarrow \Delta ABH_1 \cong \Delta BAH_2$ (by ASA).
$\Rightarrow AH_1 = BH_2$ (as corresponding sides).

b) (Fig. 84b) We consider ΔABM_1 and ΔBAM_2 | AB common side; $\angle CAB = \angle CBA$ (given); $AM_2 = BM_1 = \frac{1}{2}b$

$\Rightarrow \Delta ABM_1 \cong \Delta BAM_2$ (by SAS).
$\Rightarrow AM_1 = BM_2$ (as corresponding sides).

c) (Fig. 84c) We consider ΔABL_1 and ΔBAL_2 | AB common side; $\angle CAB = \angle CBA$ (given); $\angle 1 = \angle 2 \left(= \frac{1}{2}\alpha\right)$

$\Rightarrow \Delta ABL_1 \cong \Delta BAL_2$ (by ASA).
$\Rightarrow AL_1 = BL_2$ (as corresponding sides).

If in the solution of *Example 1* we consider instead an isosceles *right* or an isosceles *obtuse* triangle, the proofs will be sufficiently similar (do them independently!).

Example 2 Given is an acute isosceles ΔABC ($CA = CB$) with angle γ at the top vertex. If AL_1 and BL_2 are the angle bisectors and $AL_1 \times BL_2 = L$, find $\angle ALB$.

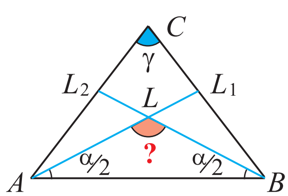


Fig. 85

Solution:

We denote the base angles of ΔABC by α: $\alpha = \frac{180° - \gamma}{2} = 90 - \frac{\gamma}{2}$.

In ΔABL the base angles AB are $\frac{\alpha}{2}$.

Thus, $\angle ALB = 180° - \left(\frac{\alpha}{2} + \frac{\alpha}{2}\right) = 180° - \alpha = 180° - (90° - \frac{\gamma}{2}) = 90° + \frac{\gamma}{2}$,

$\Rightarrow \angle ALB = 180° - \alpha = 90° + \frac{\gamma}{2}$.

Example 3
Basic Example

Prove that if in ΔABC (Fig. 86):

a) $h_c = m_c$, then ΔABC is isosceles with top vertex C.
b) $h_c = l_c$, then ΔABC is isosceles with top vertex C.
c) $m_c = l_c$, then ΔABC is isosceles with top vertex C.
d) s_{AC} and s_{BC} are perpendicular bisectors and $MN = M_1N_1$ as in Fig. 86d, then ΔABC is isosceles with top vertex C.

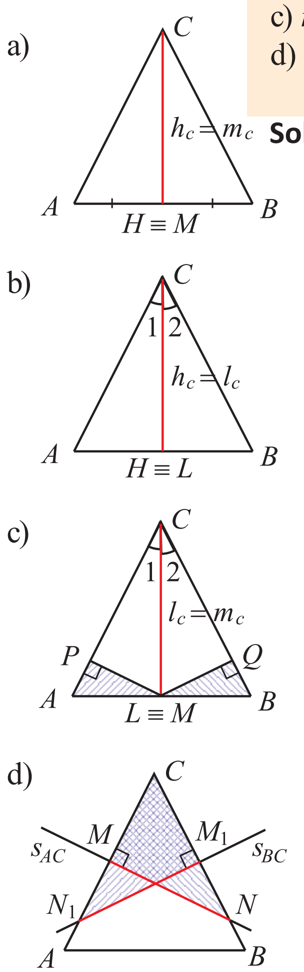


Fig. 86

Solution (Fig. 86):

a) (Fig. 86a)
$h_c \perp AB \Rightarrow \angle H = 90°$.
$h_c = m_c$ and $M \equiv H \Rightarrow AH = HB$.
$\Rightarrow \Delta AHC \cong \Delta BHC$ (by SAS) $\Rightarrow AC = BC$.
$\Rightarrow \Delta ABC$ is isosceles.

b) (Fig. 86b)
$h_c \perp AB \Rightarrow \angle H = 90°$.
l_c of $\angle C \Rightarrow \angle 1 = \angle 2$.
$\Rightarrow \Delta AHC \cong \Delta BHC$ (by ASA) $\Rightarrow AC = BC$.
$\Rightarrow \Delta ABC$ is isosceles.

c) (Fig. 86c)
From point L we drop perpendiculars LP and LQ to AC and BC, respectively.
$\Delta ALP \cong \Delta BLQ$ (by SSR) $\Rightarrow \angle A = \angle B$
$\Rightarrow \Delta ABC$ is isosceles.

d) (Fig. 86d)
$s_{AC} \Rightarrow \angle M = 90°$, and $s_{BC} \Rightarrow \angle M_1 = 90°$.
$\Delta CMN \cong \Delta CM_1N_1$ (by AAS).
$\Rightarrow CM = CM_1 = \frac{1}{2}AC = \frac{1}{2}BC \Rightarrow AC = BC$.
$\Rightarrow \Delta ABC$ is isosceles.

When solving part (c) ΔALC and ΔBLC had two corresponding sides and a corresponding angle that were equal. But there is **no** suitable criterion by which to claim that these triangles are congruent! Instead, we reason that CL is an angle bisector and point L is equidistant from the rays CA and CB of $\angle C$; i.e., $PL = QL$.

Exercises

1 Given is an isosceles ΔABC ($AC = BC$). The altitudes AH_1 and BH_2 to the legs intersect in point H. Prove that ΔABH is isosceles and $HH_1 = HH_2$.

2 In an isosceles ΔABC with base AB the medians AM_1 and BM_2 to the legs intersect in point M. Prove that ΔABM is isosceles and $MM_1 = MM_2$.

3 In an isosceles ΔABC ($AC = BC$) the angle bisectors AL_1 and BL_2 to the legs intersect in point L. Prove that ΔABL is isosceles and $LL_1 = LL_2$.

4 Given is an isosceles acute ΔABC with base AB (Fig. 87a-c).

Prove that:
a) If AH_1 and BH_2 are the altitudes to the legs, then $\Delta AH_1C \cong \Delta BH_2C$ (Fig. 87a).
b) If AM_1 and BM_2 are the medians to the legs, then $\Delta AM_1C \cong \Delta BM_2C$ (Fig. 87b).
c) If AL_1 and BL_2 are the angle bisectors to the legs, then $\Delta AL_1C \cong \Delta BL_2C$ (Fig. 87c).

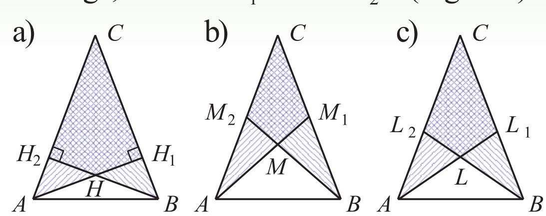


Fig. 87