

Zdravka Paskaleva, Maya Alashka

BULGARIAN MATH 1

OR **7** A TEXTBOOK

adapted for U.S. High Schools
and Excellent Middle Schools

Translated, Adapted, and Augmented
by

Zvezdelina Stankova

Founder and Director
of the Berkeley Math Circle

ΑΡΧΗ(Μ)ΕΔ
PUBLISHING HOUSE



Icons and Notation Used in the Textbook:

	Concepts (formal and informal definitions).	Basic facts.
	Important statements and formulas.	LHS = Left-hand side. RHS = Right-hand side.
	Pay attention! – Clarification to the solution of a problem.	
	Clarification of the basic facts.	
	Facts beyond the main material.	
	Historical notes and background.	

Exercises are loosely grouped by difficulty as follows:

-  **Exercises for basic proficiency (mandatory).**
-  **Exercises for high proficiency (recommended).**
-  **Exercises beyond the main material (elective).**

BULGARIAN MATH 1 (7A)

Adapted for U.S. High Schools and Excellent Middle Schools

Edition in English, 2017:

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Copyeditor: Thomas Rike, math teacher at Oakland High School.

ISBN: 978-954-779-210-4

Edition in Bulgarian, 2014:

© Publishing House “Archimedes 2” EOOD - original edition in Bulgarian, 2014.

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Referees: Prof. Dosent Todor Zheliaskov, Dosent Mariana Nenova

Printed by Alliance print – Sofia, Bulgaria

Foreword

No two people will agree entirely on all features of the “best” math curriculum, nor how it should be taught. Consequently, you need to read this introduction with an open mind, accepting that there will be differences in opinions and in approaches, knowing that the “error” in how much the outlined philosophy differs from your views is a proof of the *seriousness of the situation with the U.S. math education*.

We all want to improve the math curriculum and the math experience, and the math proficiency, and the ultimate math success of our children. Do we want too much?

We want a **lot**: that is clear. And a **lot** has to be done in order to achieve this goal: a lot of sacrifices, hard work, and compromises between views of various worlds. For the U.S. is a land where cultures mix (or don’t mix but co-exist), it is a land of opportunity, and a land of contradictions and extremes.

Our children come first: **before** our self-centered goals for self-improvement and well-being, **before** ourselves. Sacrificing personal comfort and personal self to the benefit of one’s students (or one’s children) is a sign of a great teacher (and a great parent).

When something powerful comes along and rings true, along with the listeners will come the frightened ones, who will be afraid to change, who will be scared to death to make the leap and try something new in the U.S.: something that has worked for a century elsewhere.

And such people will raise angry voices of distrust and will try to stamp out anything tender that has not yet taken root. But without such angry voices there will be no reality check, no feel for the seriousness of the situation.

The following introduction addresses a **typical** situation in a U.S. school. Exceptions do exist. There is no point in starting with this textbook series:

- unless you are ready to be completely *honest, goal-oriented, and selfless* as a teacher, a parent, or an administrator;
- unless you are ready – instead of trying to squash something new because it differs from what you are used to – to *embrace a philosophical viewpoint and a practical attitude that have proven tremendously successful elsewhere* and that have been *adapted to U.S. reality and tested on a pilot school in the San Francisco Bay Area*.

Thus, we hope that you take this new math curriculum seriously and enjoy it too.

Mathematics is a powerful universal language. Everyone needs to and can own it to the extent of his/her abilities and potential. Let this new math curriculum be a big step for the students and community at your middle and high school towards reaching this goal.

Zvezdelina Stankova
Ph.D., Harvard University

Translator, Adapter, and Co-Author
Teaching Professor of Mathematics, UC Berkeley
Berkeley Math Circle Founder & Director
Deborah and Franklin Tepper Haimo Award
for Distinguished College/University Teachers
Princeton’s Review “300 Best Professors” in the U.S.

From the education battlefield:

"You have done a masterful job of adapting this curriculum to the U.S. reality.

The U.S. Education System needs this kind of prodding. The Common Core ideas are heading in the right direction but the instruments of change are too unwieldy and we cannot just stand by and wait to see what will happen.

I hope that your efforts will be successful. Thanks for all of your hard work."

Tom Rike
Veteran math teacher
Oakland High School

Copyeditor of the Textbook Series

Reviews of the Program by Teachers, Parents, and Educators

The New Program has attracted fans and followers across the U.S., even before Pilot Year 1 was over.

We received many requests for review copies of the curriculum by schools and teachers, and by many parents, especially in the San Francisco Bay Area and in California in general.

Dr. Oleg Gleizer is the Math Department Chair at *Geffen Academy*, a new pilot middle/high school at UCLA (www.GeffenAcademy.ucla.edu). His goal there is *to create and teach a math curriculum that prepares students for the challenges of the digitized era and showcases the infinite beauty of mathematics*. Under his tutelage, Geffen Academy has adopted this textbook series for its math program and plans to start using it in the fall of 2017.

Dr. Oleg Gleizer graduated with honors from Moscow State University and received his Ph.D. from Northwestern University. He and a student of his have created an algorithm that diagnoses Parkinson's disease in its early stages with nearly 100% accuracy. He has worked for a long time as a math instructor at UCLA and as *the Chief Curriculum Developer and Lead Instructor at the Los Angeles Math Circle (LAMC)**, where he has trained nine multiple state and national champions in all sorts of Math competitions such as Math Kangaroo, American Mathematics Competitions, and others.

"I was educated in Russia, but live, work, and raise my children in the U.S.

To give my children a proper education in math and sciences, I had to leave my regular academic job and start teaching my kids myself - this is how I ended up working for the Los Angeles Math Circle. The reason was that all the U.S. math textbooks and workbooks I saw were much worse than the ones I myself had used in Russia as a child.

The 6A-8C textbooks (or Math 1-5 series) by Dr. Zvezdelina Stankova are a clear exception.

- *On the one hand, they preserve the depth and clarity of the Eastern European school of Math education.*
- *On the other hand, they are tailored to fit the needs of the US children coming out of the U.S. 5th grade in its current state. They are based on the thorough knowledge of the US pre-6th grade curriculum, including the Common Core standards.*

I took a very long and thorough look at the books and found nothing I could improve. The books are a golden standard that should be implemented country-wide. I strongly recommend them to any parent, school, and school district that want to give their children a proper education in Mathematics."

Dr Oleg Gleizer

As was quickly recognized by our pilot teachers, the Bulgarian Math Program can be successfully used:

- in middle schools, as well as, and very effectively,
- in the first 2 years of high schools.

Dr. Norm Prokyp joined the faculty at College Preparatory School in Oakland, CA (the preeminent high school in the SF East Bay Area) in 2003. He has taught at the Berkeley Math Circle and the BMC Summer Program, and very notably, he has been the Geometry instructor for the 7A-7B and 8A-8B groups at the "Math Taught the Right Way" Program at UC Berkeley.

Dr. Norm Prokyp graduated from Cornell University and received his Ph.D. from the University of Rhode Island at Kingston, studying nonlinear difference equations that model two-stage "healthy-infected" populations, suggested by epidemiologists. He has taught math ever since, working with students from grade 6 through college, and covering a wide range of subjects, from algebra to ordinary differential equations.

"How powerful is the Bulgarian approach to algebra and geometry?"

It surpasses the traditional American approach in two key ways:

- First, Bulgarian Math *presents algebra and geometry simultaneously over three years, guaranteeing much better retention* than students can get from the traditional sequence: Pre-algebra, Algebra 1, Geometry, and Algebra 2.
- Second, the Bulgarian approach is *more thorough*. Although the authors didn't have the American Common Core standards in mind, *the texts work exceptionally well with Common Core*. Students who learn algebra and geometry from these texts *develop superior skills in problem-solving, logic and reasoning, algebraic manipulation, and computation*.

How does my math background relate to the Bulgarian Math texts? I grew up in northern Virginia. My family moved several times during my education, which meant that I changed schools a lot. Although my early math education came in pieces, it worked for me. I had good teachers, many of whom cared enough about me to get angry when I wasn't doing my best. Growing up, math was my favorite subject, so I always started my math homework first.

- From engineering school, I learned *the value of computation, precision, and fluency of execution*.
- From mathematics, I learned that *to really understand a subject, you must be able to take it apart and then put it back together*.

These values resound throughout the Bulgarian curriculum.

All my favorite math problems involve pictures, so I consider myself a geometer at heart.

- *Solid geometry, vector geometry, plane transformations, and other topics that are often marginalized in an American curriculum play a central role in the Bulgarian texts. Hence, the Bulgarian curriculum is ideal training for future physicists, engineers, and computer scientists.*
- *Students emerge from this program ready to study analytic geometry and functions. Thus, this curriculum would be a superior choice for students from all grades 6 to 10.*

I use the Bulgarian curriculum for my geometry classes at the "Math Taught the Right Way" program at U.C. Berkeley. My students, who range from grade 7 to 10, have embraced this curriculum.

- *I love teaching from these texts, and I love sharing them with other teachers.*
- *I eagerly await each chapter as it emerges, newly translated into English, and*
- *I'll be first in line to buy each new volume as it becomes available.*

The miracle of the Bulgarian Math Program is that it works exceptionally well with a group of mature and industrious younger students. Indeed, the purpose of the Bulgarian texts is to provide the best possible training for tomorrow's mathematicians and scientists.

Dr. Norm Prokup

Here follow reviews of the Bulgarian Math curriculum by two more instructors, Elysee W.-Egolf and Ellen Kulinsky, at the 6th grade Algebra, Geometry, and Problem Solving classes at the “Math Taught the Right Way” (MTRW) Program at UC Berkeley.

I grew up in the U.S. with textbooks ...

... that were rote and unmotivated, with no applications for algebra students, no geometry or number theory for 6th graders, and no guidelines in writing mathematics. The curricula that are popular in the Bay Area at the time of writing have sought to resolve this issue by making the entirety of student education exploratory, without practice or summarization of what is expected - and in both paradigms, the textbook authors take dozens of pages to get to the point.

The current adapted Archimedes curriculum, by contrast, gives examples for students to explore and then guides them to correct conclusions. Algorithms are offered only after clarifying their logic and students are taught to work clearly and thoroughly through both numerical exercises and problems requiring more complex reasoning - and all this while calmly weaning reliance from calculators. It picks up where books like Singapore Math leave off, and offers students of all backgrounds and abilities entry into the deeper understanding they require to master future material.

Because the topics are covered so thoroughly and a workbook is included with extra problems, there is absolutely no need for teachers to create extra worksheets, which means that the teacher can focus on planning lessons and evaluating student work. The Bulgarian Math textbook is more relaxing for both student and teacher to work from.

Elysee Wilson-Egolf

BA in Mathematics, UC Berkeley

Bentley High School Math Teacher
Instructor at MTRW and BMC

These texts are such an improvement from ...

... the typical textbooks used in the American math education system. Rather than merely presenting formulas, these materials focus on their derivation and various applications. The result is remarkable, studying math becomes engaging, an exploration for the students, rather than the tedious memorization and monotonous plug and chug students may be used to in the American math curriculum.

The books contain beautiful explanations and plenty of examples, easing students into challenging problems. And although the books cover a wide range of mathematical subfields (geometry, algebra, etc.) the students are able to see how concepts from different subfields are integrated, as they are in the real world, outside the classroom.

In my personal experience, growing up, I attended a Math Circle, which exposed me to creative, thought-provoking problem-solving approaches, as well as the connections between mathematical subfields, so I knew that the math classes I was taking in middle and high school were not a fair representation of how enjoyable studying math could be. These are the first math books I have come across that bridge the gap and incorporate the depth, clarity, and excitement of a Math Circle, while also tailored to the American math standards.

These materials are the perfect tool for students and math teachers alike to truly experience and share the beauty and joy of studying mathematics.

Ellen Kulinsky

Math Major, UC Berkeley
Instructor at MTRW and BMC

The quotes on this page are from more pilot teachers, parents, and educators. You will come across them in the Introduction chapter, in their original context. Thus, we strongly encourage you to carefully read the Introduction first, before continuing with the actual math material.

"This book has been a tremendous resource for my students, and I wish I had it when I was in school. The lessons are laid out in a very clear manner. The math is approachable or challenging to meet the needs of a child at any level. There are many math books that claim to reach every person. This book successfully does so, and it is one of the best."

Natalya St. Clair, pilot teacher
co-author, "The Art of Mental Calculation"

"I feel that the Bulgarian texts are much more thorough than the American textbooks I am using for 7th grade. The truth is that had there been this year also a translated 7th grade Bulgarian textbook I could use, I would have definitely used it for all of my classes. It is much easier to teach from and saves me work."

George Khasin, pilot teacher

"The Bulgarian curriculum is more rigorous than others I have used as it introduces Algebra in early grades. Variables are integrated into daily lessons so that students become as familiar with manipulating equations with x and y as they do with real numbers. This will pave the way for an easy transition to Algebra and set them up for success in higher grades."

Laurie Gold, pilot teacher

"Our daughter has ...
... learned to love math with the Bulgarian math curriculum. It stimulates not only her technical mathematical abilities but also challenges her to strategically and creatively solve problems."

This program keeps her interested and excited."

Eve Maidenberg

6th grade pilot parent

"Thank you so much...
...for this feedback! Our daughter is struggling to catch up. I feel like she is learning more math now in one month than she did all last year. Thank you for your help!"

6th grade pilot parent

"As a teacher and teacher educator, I am familiar with several ...
... math programs that have been used throughout the past 20 years in California. The Bulgarian math program is exceptional in the way it supports students' deep understanding of mathematical concepts, develops students' ability to transfer skills to new and unfamiliar mathematical situations, and creates a mathematical language for students to express themselves mathematically, using numbers, drawings, and narratives."

Teaching is sort of like farming. You never know how your crops are going to turn out, but I have noticed with my son that his work has improved, and I really like the way he is talking about math. I can see, and most importantly hear, the way my child interacts with the Bulgarian math program and I am impressed with the strong foundation it is giving him that will surely lead to lifelong competency and the ability to pursue more advanced mathematics."

Jacqueline Regev, 6th grade pilot parent and educator

"This Middle and High School Math Program is deeply aligned with the Common Core Mathematics Standards".

Dr. Phil Daro

One of the Three Principal Writers
of the Common Core
Mathematics Standards

Contents

Foreword	III
Reviews of the Program	IV
Contents	VIII
Introduction to the Program	X
1. Mission Impossible. 2. The Find	X
3. Origins. 4. Purpose	XI
5. The Temple of Mathematics	XII
6. Relationship with Mathematics	XIV
7. A Cultural Shift.....	XV
8. The Pillars of Mathematics.....	XVI
8.1. Algebra. 8.2. The Missing Algebraic Link	XVI
8.3. The Geometry Neverland	XVIII
8.4. Golden Geometry Days of My Childhood	XIX
8.5. A Reality Check: Geo-Algebra in the U.S.?	XX
Math Curriculum at a Glance.....	XX
8.6. The Main Principles	XXI
9. Differentiate and Scaffold	XXI
9.1. Examples of Scaffolding.....	XXII
9.2. What is the Math 5/8C Textbook?.....	XXII
9.3. Does Scaffolding Really Work?	XXII
10. The Value of a Teacher	XXIII
11. Can a U.S. School Succeed w/ the Program? XXV	
11.1. The U.S. Gene Pool.....	XXV
11.2. Can Girls Succeed in Math?.....	XXV
11.3. The Needed Cultural Shift	XXVI
11.4. Continuity of the Curriculum.....	XXVI
11.5. Local Support	XXVI
12. Within the Global U.S. Picture.....	XXVII
13. Math 1/7A Program by Topics/Lessons XXVIII	
14. How to Work with Math1/7A Materials..	XXX
15. Credits	XXX
16. Math Taught the Right Way.....	XXXI
17. Acknowledgments.....	XXXII

Let the Math Begin!

Chapter 1. Initial Review.....	1
1. Fractions, Parts, and Percents (Review)	2
2. Rational Numbers and Powers (Review).....	4
3. Circles and Polygons (Review).....	6
4. Graphing and Data (Review)	8
5. How Mathematicians Write Solutions	10
6. Entrance Level: General Test.....	12
7. How to Study for and How to Take Tests	14

Chapter 2. Polyhedral Solids.....	15
8. Prism. Regular Prism	16
9. Right prism. Exercises	18
10. Surface area of a Right Prism	20
11. Surface area of a Right Prism. Exercises	22
12. Volume of a Right Prism.....	24
13. Volume and Surface Area of a Right Prism. Exercises	26
14. Pyramid. Regular Pyramid.....	28
15. Regular Pyramid. Exercises	30
16. Surface Area of a Regular Pyramid	32
17. Surface Area of a Regular Pyramid. Exercises	34
18. Constructing Models of Polyhedral Solids. Practical Exercises	36
19. Volume of a Regular Pyramid.....	38
20. Volume and Surface Area of a Regular Pyramid. Exercises.....	40
21. Summary of the Topic “Polyhedral Solids” ...	42
22. Tests on the Topic “Polyhedral Solids”	44
Chapter 3. Round Solids	45
23. Right Circular Cylinder.....	46
24. Surface Area of a Cylinder.....	48
25. Volume of a Cylinder	50
26. Right Circular Cone	52
27. Surface Area of a Cone	54
28. Surface Area of a Cone. Exercises	56
29. Volume of a Cone.....	58
30. Sphere. Surface Area of a Sphere	60
31. Ball. Volume of a Ball.....	62
32. Surface Area/Volume of a Ball. Exercises.....	64
33. Round Solids. Practical Exercises.....	66
34. Summary of the Topic: “Round Solids”	68
35. Test on the Topic: “Round Solids”	70
Chapter 4. Proportions	71
36. Ratio. Proportion.....	72
37. Proportionality. Constant of Variation	74
38. Basic Property of Proportions.....	76
39. Properties of Proportions	78
40. Applications of Proportions. Exercises	80
41. The Ratio $a : b : c$. Exercises	82
42. Presenting Data via a Circular Diagram	84
43. Interpreting Data in a Circular Diagram	86
44. Summary of the Topic “Proportions”.....	88
45. Test on the Topic “Proportions”	90

Chapter 5. Integer Expressions. Polynomials	91
46. Rational Expressions. Introduction	92
47. Integer Expressions. Numerical Value	94
48. Monomial. Standard Form of a Monomial.....	96
49. Adding and Subtracting Monomials. Similar Monomials.....	98
50. Adding and Subtracting Similar Monomials. Exercises	100
51. Multiplying, Taking Powers of, and Dividing Monomials	102
52. Polynomials. Standard Form.....	104
53. Adding and Subtracting Polynomials	106
54. Multiplying a Monomial w/ a Polynomial...	108
55. Multiplying a Polynomial w/ a Polynomial ..	110
56. Multiplying Polynomials. Exercises	112
57. Summary of the Topic “Integer Expressions” .	114
58. Test on the Topic “Integer Expressions”	116
Chapter 6. Algebra on Polynomials. Part I	
Degree 2 Formulas	117
59. Equivalent Expressions	118
60. Formulas $(a \pm b)^2 = a^2 \pm 2ab + b^2$	120
61. Formulas $(a \pm b)^2 = a^2 \pm 2ab + b^2$. Exercises..	122
62. Formula $(a + b)(a - b) = a^2 - b^2$	124
63. Formulas for Quick Multiplication. Exercises .	126
64. Quick Multiplication. Applications.....	128
65. Factoring Polynomials w/ a Common Factor .	130
66. Factoring by Using the Formulas for Quick Multiplication	132
67. Factoring by Grouping	134
68. Factoring by Combining Various Methods ...	136
69. Factoring by Various Methods. Continuation. .	138
70. Equivalence Transformation. Applications...	140
71. Summary of the Topic “Algebra on Polynomials, Part I”	142
72. Test on the Topic “Algebra on Polynomials I”	144
Chapter 6. Algebra on Polynomials. Part II	
Degree 3 Formulas (Optional)	145
73. The Formulas $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$..	146
74. The Formulas $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$..	148
75. Formulas for Quick Multiplication. Exercises with Degrees 2 and 3.....	150
76. Factoring by Using the Formulas for Quick Multiplication	152
77. Short Summary of the Topic “Algebra on Polynomials, Part II”	154

Appendix 1: Algebra	XXXIII
Appendix 2: Geometry	XXXIV
Biographical Sketches	XXXVI
Answers	XXXVIII

Projects: Cheat Sheets. Summary Tables

Chapter 1. Initial Review.

- Properties and operations w/ fractions
- Six basic types of equations w/ parts and %
- Properties and operations w/ rational numbers
- Properties of powers with integral exponents
- Notation and formulas for circles and polygons
- Data representations; conversions; average
- CoCoClear Rubric. Examples of scoring

Chapter 2. Polyhedral Solids

- Notation/formulas for prisms and pyramids
- Word problems on polyhedral solids. Examples

Chapter 3. Round Solids

- Notation/formulas for cylinders, cones, spheres
- Word problems on rounds solids. Examples

Chapter 4. Proportions

- Concepts of ratio, proportion, extended ratio, directly proportional, constant of proportionality
- Properties of proportions: formulas, examples.

Chapter 5. Integer Expressions. Polynomials

- Concepts of monomial, polynomial, degree, coefficients, similar monomials, standard form
- Properties and operations with polynomials

Chapter 6. Algebra on Polynomials. Part I

- Concepts of equivalent expressions, identities
- Degree 2 formulas for quick multiplication
- Degree 2 formulas for factoring
- Methods for factoring polynomials:
 - by factoring a common factor
 - by grouping
 - by using the formulas for factoring
 - by combining methods

Chapter 6. Algebra on Polynomials. Part II

- Degree 3 formulas for quick multiplication
- Degree 3 formulas for factoring
- Summary of formulas for:
 - quick multiplication of polynomials
 - factoring of polynomials

Introduction to the Program

1. Mission Impossible

Every year, for the past 20 years, while working* with 250 students in grades 5-12 at the Berkeley Math Circle (BMC), I have been asked over and over again by parents, students, and teachers alike:

*"Can you give us good textbooks for our middle and high school math curriculum?
What we have is not working and it hasn't worked for years!"*

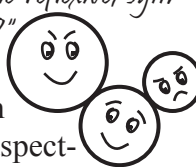


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And for these 20 years I could think of only two sources that I would also give to my own children: my *old Bulgarian middle school textbooks*, and the *Art of Problem Solving books (AoPS)*. But **neither of these choices would have worked**, despite the fact that they are, in my opinion, among the best middle school textbooks in the world. Indeed, they are written for *advanced* audiences as measured by U.S. standards:

The very first sentence in my old *Bulgarian 6th Grade Algebra* textbook would stump many of my freshman college students: "Since we have worked [in 5th grade] with the set of rational numbers $\mathbb{Q}$ and are familiar with its properties, we move on." And without further ado, Lesson 9 went on to studying the subtle differences between functions and relations!

My old *Bulgarian 6th Grade Geometry* textbook did not lag behind. On page 42 it asked: "Is the relation 'is tangent to' on the set of circles in the plane reflexive, symmetric, or transitive?" Just to explain what the question means to an unsuspecting, typical 6th grader would take a whole lot of time!



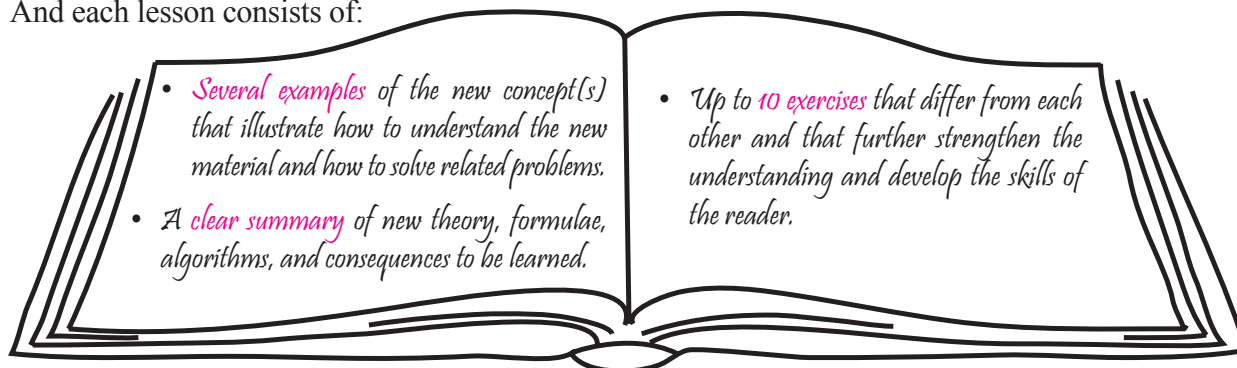
Although the *AoPS books* are a wonderful resource, they are not designed as "textbooks" in a regular math class environment. I tend to recommend these books for my more advanced Math Circle students, who are guaranteed to have extra support outside of a typical school classroom.

And so, short of designing a new program myself, I was helpless to help anyone seeking a better middle school math curriculum....

2. The Find

....That is, until I asked my mom to send me some recent Bulgarian math textbooks, to teach my own kids in the U.S. She quickly found the most popular textbooks in Bulgaria: the present series from "Archimedes." When I opened them, I was immediately taken by their main feature: *Each lesson is on exactly 2 pages!*

And each lesson consists of:



* I am a Professor of Mathematics at UC Berkeley. I am originally from Bulgaria, which has a rigorous mathematics curriculum. I founded BMC to help pass the math enthusiasm on to the younger generation in the San Francisco Bay Area.

The textbooks are written in a *straightforward* and *simple manner*, capturing the *elegance* and *robustness of mathematics* itself. There is no “hide-and-seek”:

- everything is *crystal clear*;
- *one look* at a lesson suffices to see what it will teach you;



- there is no need to look around on the Internet for other sources: the *textbook*, *workbook* and *problem collection* are all one needs for a *successful course* in middle school mathematics.

A pilot 6th grade teacher shares:

"I feel that the Bulgarian textbooks are much more thorough than the American textbooks I am using for 7th grade. The truth is that had there been this year also a translated 7th grade Bulgarian textbook I could use, I would have definitely used it for all of my classes. It is much easier to teach from and saves me work."

George Khasin, teacher

3. Origins

This middle school math program is based on one of the leading programs of today in Eastern Europe, developed and implemented by the well-known Bulgarian mathematician and educator Professor Georgi Paskalev. His relatively new mathematical program targeting specifically middle and high schools in Bulgaria has conquered 60-70% of the Bulgarian education market during the single decade since its conception.

The new program:

- has *partially departed* from the traditional “old school” Bulgarian program, and
- has *crossed boundaries* to meet the standards and demands of a global educational market, such as in the U.S.; yet,
- has *preserved the intrinsic pedagogical* and *mathematical values* of the old curriculum.

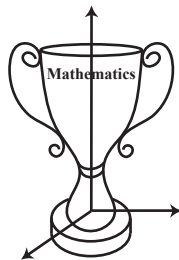
Quite appropriately, the company founded by Professor Paskalev is called “*Archimedes*”: its publications live up to the name of the great Greek mathematician, scientist, and inventor from antiquity.*



4. Purpose

The present math program extends the mathematics taught in elementary school and completes it to a comprehensive K-8 math program that prepares students:

- to enter any high school with *confidence* in their math skills and knowledge;



- to be *successful* at any level of high school math classes;

- to develop *maturity* and a *life-long relationship* with mathematics that:
 - will distinguish them among their peers;
 - will serve them well in any profession and in any endeavor in which they chose to engage.

* Archimedes' picture is from Wikipedia: <https://la.wikipedia.org/wiki/Archimedes#/media/File:Archimedes1.jpg>.

5. The Temple of Mathematics

The math education of every child is like a *building*; every year we hope that a new floor emerges on top of the previous floor. Unfortunately, more often than not, our children come *under damaging* (even though, perhaps, well-intended) *actions of math educators on any floor of this building*. For example,

- What happens when a teacher decides to spend 3 months reviewing single-digit multiplication in 4th grade?

Well, the 4th graders become really skilled with their multiplication tables, and, say,

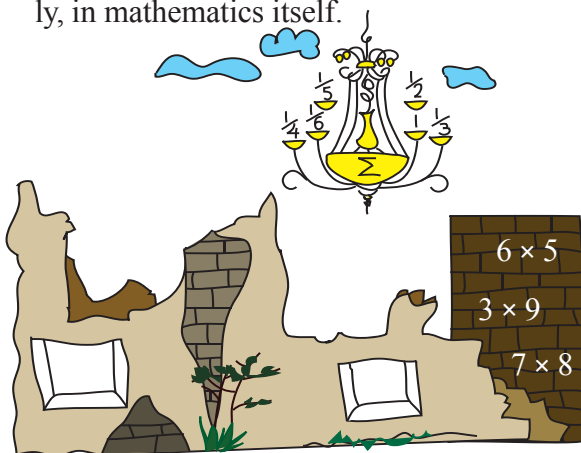
- a 6-foot thick wall of “single-digit multiplication” is erected on the 4th floor of the building.

Alas, time is not stretchable, and such educational decisions inevitably cause:

- some other wall on that same floor to be thin or non-existent; e.g., a “wall” of word problems, logical thinking, or a rich geometric vocabulary.

As a result, when in 5th grade the next teacher tries to build “walls” upon such previously expected but non-existent or weak walls, not surprisingly,

- the structure collapses! And in the ensuing debris, time is wasted figuring out what and why it happened. Worse, a student may lose confidence in his/her math skills, in the teacher’s abilities, in the integrity of the math program, and, sadly but not infrequently, in mathematics itself.



What may strike us as even *more dangerous* for the young developing mind (yet, still well-intended) is the desire of many textbooks and some teachers nowadays (perhaps, under pressure from school administration or for other reasons) to occasionally show off via:

- *extra-curricular activities* that are unsupported by and incompatible with the math building erected up to that moment. A typical example is an “enrichment” section on:
 - *Series!* This topic, ordinarily covered in a college Calculus course, has made its debut in standard, well-known U.S. 6th grade textbooks, alongside computer codes that intend to demonstrate the “growth” of the “partial sums of the series”.

Let us reason logically here.

How can one *truly understand* why the sum

$$1 + 1/2 + 1/3 + 1/4 + 1/5 + \dots$$

adds up to infinity, but the sum

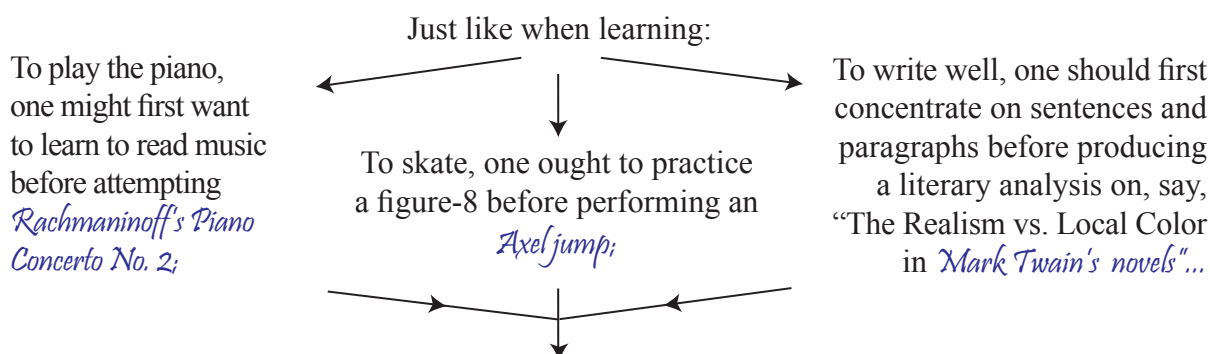
$$1 + 1/2 + 1/4 + 1/8 + 1/16 + \dots$$

does not, *before* having mastered, say, the *distributivity laws*?! Although U.S. students may be able to recite from memory these distributive laws (e.g., $a \cdot (b + c) = a \cdot b + a \cdot c$), by and large, they will have no clue when, why, and how to apply these laws, so much more efficiently.

Thus, we ask again: what effect will such a “fancy” and untimely introduction of high-level math ideas have on the budding young mind? To put it bluntly, the situation will be equivalent to hanging a fancy chandelier on a non-existent ceiling. Imagine the “aftermath” of such an action, the loss of time, energy, and confidence.

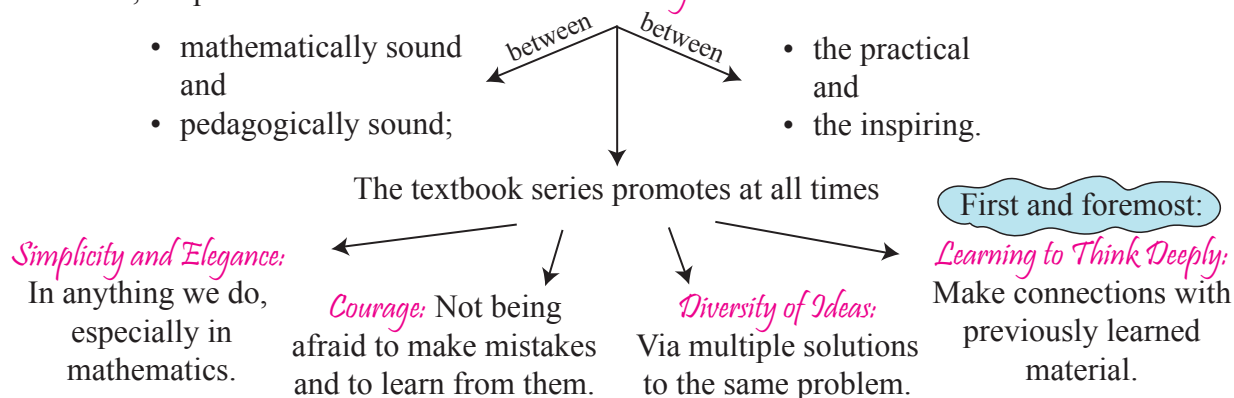
And we are **not** talking about the precocious youngster who can handle with ease and excitement any bombardment with advanced math material, whether on series, in combinatorics, abstract algebra, or other such topics! We have plenty of such young people at the Berkeley Math Circle and observe this phenomenon on a weekly basis all the time.

- We are talking about the *typical* U.S. student, whose primary or only source of learning math is through math classes in school. →
- The math structure for this student will be *jeopardized* by such disorderly and out-of-context teaching.



... So should there be *order*, *balance*, and *harmony* in everything we do, teach, and learn. We **cannot** lower our standards when teaching mathematics to the young minds. On the contrary, we must be especially careful because of the *hierarchical structure of mathematics*.

And thus, the present textbook series* has found the *golden ratios*:



These are not just catchy, “politically correct” phrases intended to sell more copies of this textbook. They truthfully describe this math program and help build a solid math structure, with adjoining “walls, windows, and doors,” through which to move from topic to topic and make connections between various concepts, formulas, and ideas.

For someone who has not worked deeply with mathematics before, the words *love*, *creativity*, and *excitement* are usually light years away from having anything to do with math. Yet, here they all are, in a testimony of a pilot 6th grade parent, leading us to the next topic – *how a student interacts with and feels about mathematics*.

* The pun was not intended in the former, but it was intended in the latter instance! ☺

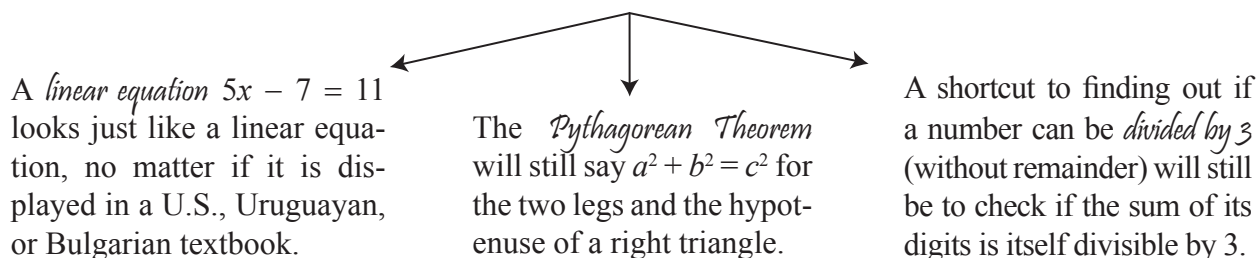
☺ A pilot 6th grade parent shares:

“Our daughter has learned to love math with the Bulgarian math curriculum. It stimulates not only her technical mathematical abilities but also challenges her to strategically and creatively solve problems. This curriculum keeps her interested and excited.”

☺ Eve Maidenberg, parent

6. Relationship with Mathematics

With some notable exceptions, especially in Geometry, the actual math contents in these textbooks does not “look” much different from just about any standard U.S. middle school textbook. After all,

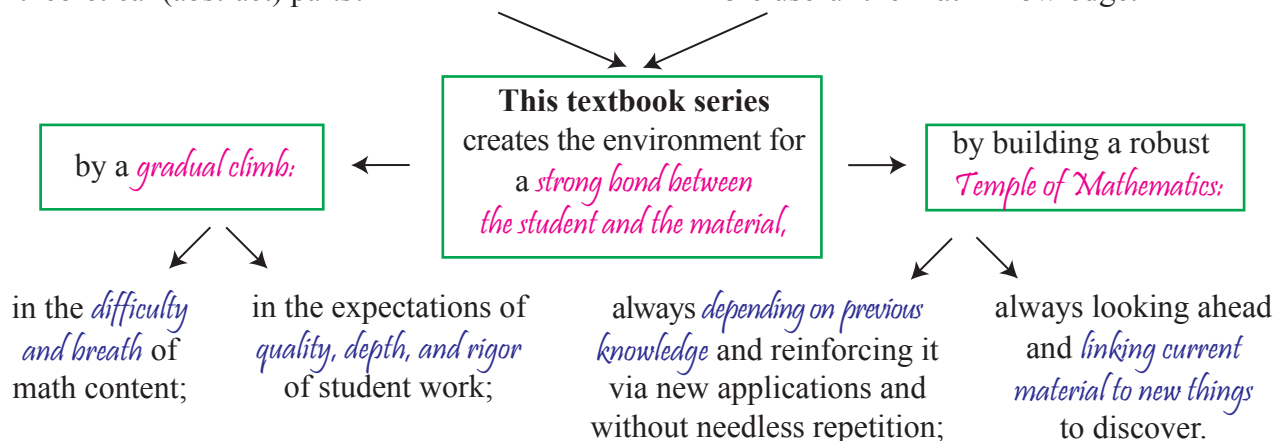


... because *mathematical truth is universal*.

Beyond the contents, two *fundamental questions* that a good teacher must ask about any textbook are:

Question 1: How are the math facts *presented*? In what order and what style? Is there a balance between computational (algorithmic) parts and theoretical (abstract) parts?

Question 2: What *relationship* between the student and the material is developed? The deeper the relationship, the more lasting and more useful the math knowledge.



At any time throughout the school year the place and importance of the concepts, ideas, and techniques studied are *clear within the "bigger picture" of Mathematics*. Unfortunately, just as described on the previous pages, meddling with textbooks and curricula is a wide-spread phenomenon in the U.S. middle school math education! Here are four sure ways to ruin the structure and purpose of any math textbook, whether good or bad:

1. Omitting sections.
2. Changing the order of the material in the textbook.
3. Cluttering lessons with hand-outs from other curricula.
4. Sacrificing the depth and teaching to the answers!...



Remember that no one wants to climb up a flimsy structure! As a result, over time the top floors of the Math Temple will become uninhabitable if the actions in the red box run amok in math lessons.

Instead, the deep thought and math wisdom put into creating these textbooks must be respected, or there is no point in starting with this program. Thus, the present textbooks must be *followed just as systematically* as one studies a new language, violin, or chess.

7. A Cultural Shift

Several features of the textbooks, which are typically ignored or de-emphasized in standard U.S. middle school curricula, represent a *cultural shift* in how mathematics is viewed and studied in the U.S., but which are very well-known around the world and have been adopted by many countries for decades, if not centuries.

7.1. Reading Mathematics. Every math problem has “words” – these are concepts expressed in different ways, whether as numbers, standard everyday words, or by visual aids such as diagrams and figures. Emphasizing the different forms of “words” as a regular part of math language and math communication and *learning to read, interpret, and write in “words”* is a major part of the new curriculum. The pilot parent’s comments below addresses this and more:

A pilot 6th grade parent shares, 7 months after the new curriculum was launched:

“As a teacher and teacher educator, I am familiar with several math programs that have been used throughout the past twenty years in California. The Bulgarian math program is exceptional in the way it supports students’ deep understanding of mathematical concepts, develops students’ ability to transfer skills to new and unfamiliar mathematical situations, and creates a mathematical language for students to express themselves mathematically, using numbers, drawings, and narratives.”

I can see, and most importantly hear, the way my child interacts with the Bulgarian math program and I am impressed with the strong foundation it is giving him that will surely lead to lifelong competency and the ability to pursue more advanced mathematics.

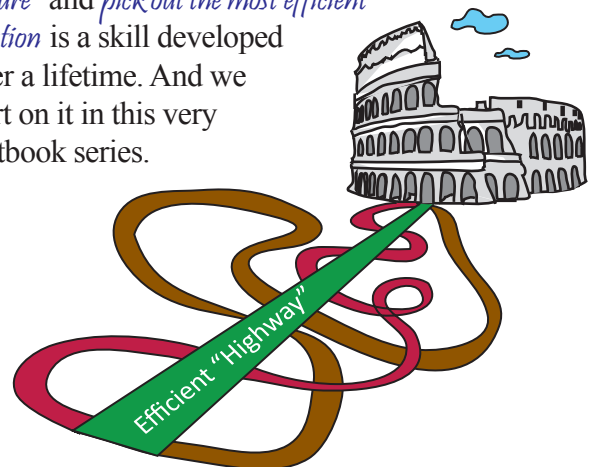
Jacqueline Regev, parent and educator

7.2. Logic and Communication. Learning to know *correct from flawed reasoning* and being *able to explain* one’s mathematical ideas smoothly and convincingly to others will represent a gradual move towards *a more mature relationship with mathematics*.

7.3. Writing Mathematics. “Showing your work” is only the beginning to understanding how math really works. All problems have answers in the back of the textbook. Bringing back just an answer on a homework problem from the textbook will be worth no credit. *The explanation that leads to the answer* will be what counts in our study of middle school mathematics. Even harder than learning to correctly interpret problems is learning to consistently *write solutions in a correct, complete, and clear way*. (See Lesson 5 in the 6A textbook.)

7.4. Multiple Solutions. Although there is usually (but not always!) only *one* correct answer to a math problem, the beauty of mathematics is that there may be *different solutions leading to that answer*. These are not some “subjective opinions”; rather, they are objective mathematical creations that obey the laws of Logic. Students will learn that each solution (and even each incorrect attempt) has its value and usefulness in the long run and that one needs to *open up to others’ ideas and ways of thinking* as a path to enriching oneself, to becoming more proficient and, ultimately, wiser.

7.5. Efficient Solutions. Yet, among “all the roads [that] lead to Rome,” there might be a shortest or an easiest to follow. The ability to *see “the big picture”* and *pick out the most efficient solution* is a skill developed over a lifetime. And we start on it in this very textbook series.



8. The Pillars of Mathematics

There are *three pillars* of mathematics: Algebra, Geometry, and Logic/Problem Solving. They must be present at any stage of pre-college math education.

Middle school is the time when math clicks: when the young mind can handle logical leaps and sophisticated reasoning. Yet, instead of being ready for the “big moment,” when middle school knocks on the door, teachers and parents alike start:

8.1. Algebra*

There is so much fuss to distinguish between Pre-Algebra, Algebra 1, and Algebra 2... when, in fact, there is no such thing as “Pre-Algebra” as a math subject, nor are there “Algebra 1 or 2”! In a well-designed and well-executed K-12 program,

- *Arithmetic*
↓ gradually turns into
- *Algebra* $\xrightarrow{\text{naturally leads to}}$ • *Real Analysis.***

Alas, the typical U.S. student:

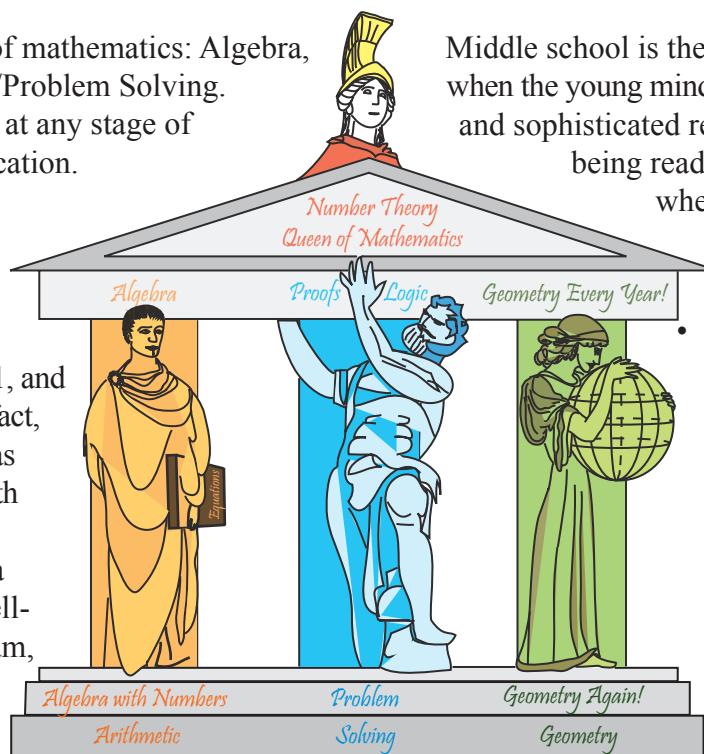
- hops along uneven and vaguely defined passages between Pre-Algebra, Algebra 1, Algebra 2, Pre-Calculus, and Calculus;
- experiences a chop-chop approach of poorly matched Algebra curricula from grade to grade;
- is taught to be more concerned about memorizing algorithms*** and passing test benchmarks,

than what is truly important:

- *making connections* and fitting the many pieces together in a giant “jig-saw puzzle” to see the “*big picture*”;

and little by little,

- building a *solid Pillar of Algebra* in his/her Temple of Mathematics.



- worrying about students “*missing algebra skills*”;
- pulling the class back to arithmetic operations because all of a sudden “*pre-algebra*” is too hard;
 - going in circles;
 - losing time, energy, and confidence; and
- ultimately, *weakening the math structure.*

8.2. The Missing Algebraic Link

Why is this happening? The major Algebra mishaps in U.S. middle schools are due to events that have already occurred in elementary school, and NOT necessarily due to poor arithmetic skills. Expecting that a middle school student will leap from *arithmetic operations with numbers*, e.g.,

$$97 \cdot 2016 = 195,552,$$

to *algebraic operations with symbols*, e.g.,

$$(a + b)(a - b) = a^2 - b^2,$$

is asking for trouble. There is a whole array of missing intermediate steps, to which we refer as “*Algebra with Numbers*” and which can and should be started very early in the learning process.

How early? On the next page we list some examples from *early grades in Bulgaria*. They represent well what happens in the algebraic progression also in other countries around the world.

* Algebra was present in ancient Greek mathematics mostly as *geometrical algebra*. The word “algebra” is Latin for the Arabic “al-jabr,” or “the reunion of broken parts”. In our “divine” interpretation of mathematics, the Greek mathematician *Diophantus, the Father of Algebra*, is supporting the math structure on the left and is the only non-deity in the drawing; yet, as a mortal, he had tremendous world influence in both Algebra and Number Theory.

** a.k.a. *Calculus* in first 2 years of U.S. college education.

*** Automated “recipes” for some types of problems.

Sample “Algebra with Numbers” Problems from Elementary School in Bulgaria

1st grade. Calculate in two ways $7 + 9 + 3$.

- **Expected written solution:**

$$(7 + 9) + 3 = 16 + 3 = 19;$$

$$7 + (9 + 3) = 10 + 9 = 19.$$

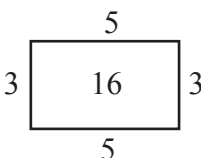
2nd grade. Fill in the blanks. Where have we seen such calculations?

$$[5 + 5] + [3 + 3] = 2 \cdot \diamond + 2 \cdot \diamond = \diamond$$

- **Expected written solution:**

$$[5 + 5] + [3 + 3] = 2 \cdot 5 + 2 \cdot 3 = 10 + 6 = 16.$$

I remember! The sides of a rectangle that are opposite are equal. The perimeter of the rectangle is twice one of the sides plus twice the other side.



3rd grade. Calculate the expression:

$$1000 - (132 \div 3) \cdot 4.$$

- **Expected written solution:**

$$1000 - (132 \div 3) \cdot 4$$

$$= 1000 - 44 \cdot 4 = 1000 - 176 = 824.$$

4th grade. Check whether the equality

$$(a + b) - c = a + (b - c)$$

is true for $a = 347$, $b = 217$, and $c = 105$.

- **Expected written solution:**

$$(a + b) - c = (347 + 217) - 105$$

$$= 564 - 105 = 459;$$

$$a + (b - c) = 347 + (217 - 105)$$

$$= 347 + 112 = 459.$$

Since we got the same answer 459, the equality is true in this case.

5th grade. Find the unknown number x if

$$8\frac{4}{5} \div x + 5.3 = 25 \cdot 0.3.$$

- **Expected written solution:**

$$8.8 \div x + 5.3 = 7.5$$

$$8.8 \div x = 2.2$$

$$8.8 \div x = 7.5 - 5.3$$

$$x = 8.8 \div 2.2 = 4.$$

This sequence of five random exercises in “Algebra with Numbers” starts in 1st grade in Bulgaria and continues all the way to 6th grade. And then *the “true” Algebra commences*, without hiccups, without drama, and with huge success, because:

- students have been *preparing for Algebra for a long time*, even if no one called it “Algebra”;
- a *continuum in studying Algebra* has provided smooth transitions at every step of the way.

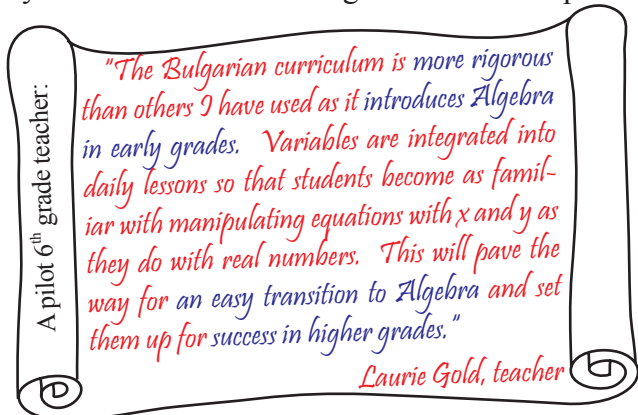
And just as playing the piano for the 6th year will likely encompass some harder, lovelier pieces, so will studying Algebra for the 6th year conquer tougher, nicer problems.

This short list of problems, culminating in the 5th-grade equation with fractions and decimals, also explains why we could NOT start our 6th graders in the U.S. on the 6th grade Bulgarian curriculum: they were NOT ready for it! We roughly estimated that they were lagging behind their Bulgarian peers by half a year in terms of algorithmic manipulations; and they were at least a year behind in algebraic skills and math maturity.

Of course, the easiest thing was to give up. But anyone could do that. We could do better. We did MUCH better! In fall 2015 we started the new curriculum in the pilot middle school:

- by first covering all of the Bulgarian 5th grade curriculum in 6th grade here; and
- only then moving to the Bulgarian 6th grade curriculum in the last third of the 6th grade here and covering half of it.

“Were you successful with the Algebra part?” you ask. Let one of the 6th grade teachers respond:



8.3. The Geometry Neverland



In our drawing of the pillars of mathematics Geometry is represented by Gaia, the Greek Goddess of Earth (“geo” = Earth in Greek). The truth is that Geometry is the “*Cinderella*,” with U.S. math education being its “step-mom”. In K-8 grades Geometry appears sporadically here and there, with few (if any) connections to the rest of the math curriculum.

Later, in U.S. high school curriculum, there is *one intense year of Geometry*, during which:

No other math courses are studied.

This in itself is a recipe for trouble: the few connections between Geometry and the rest of mathematics that were hinted before in the math curriculum are mostly forgotten during that year.

Geometry is studied in isolation!

4 years worth of Geometry are compressed in one year and they are presented to, by and large, students unprepared for such a huge single “dose” of Geometry.

Unlike making bread, there is no time for Geometry knowledge to “rise”!

Proofs are introduced through this Geometry course, often in the notorious “2-column format”. Truly powerful Geometry theorems are not reached. As a result, many students not only do **not** learn how to do proofs, but also...

They end up, by association, disliking Geometry itself!

Question: Why talk about a **high** school Geometry course in the midst of a **middle** school curriculum?

Because everything starts at an early age and builds up over the years, regardless of whether it is Algebra, Geometry, Logic, playing an instrument, or learning to write excellent essays. And if we do not have the *vision of what awaits the young learner* in a few years from now, we will be doing a big disservice to young learner, and we will be cutting off his/her chances to succeed in the future.

Knowledge and skills in Geometry cannot be amassed “miraculously” in a year. *Geometric intuition develops over time*, not instantaneously. To package and present several years worth of Geometry in one course to the untrained mind, and to top it off with introducing formal proofs (a hard topic even in a college curriculum) inevitably leads to a poor outcome.

Worst of all, by the time the high school Geometry course comes, *“the train is already gone”*: it is too late to repair the damage that was systematically done over the K-8 years to the Geometry Pillar in the Temple of Mathematics.

Consequently, the young budding math mind has *missed the creative and formative time* in middle school for a fundamental geometric development.

Insufficient Geometry knowledge and skills will reverberate throughout a person’s whole life, including college and post-college career.

The truth again:

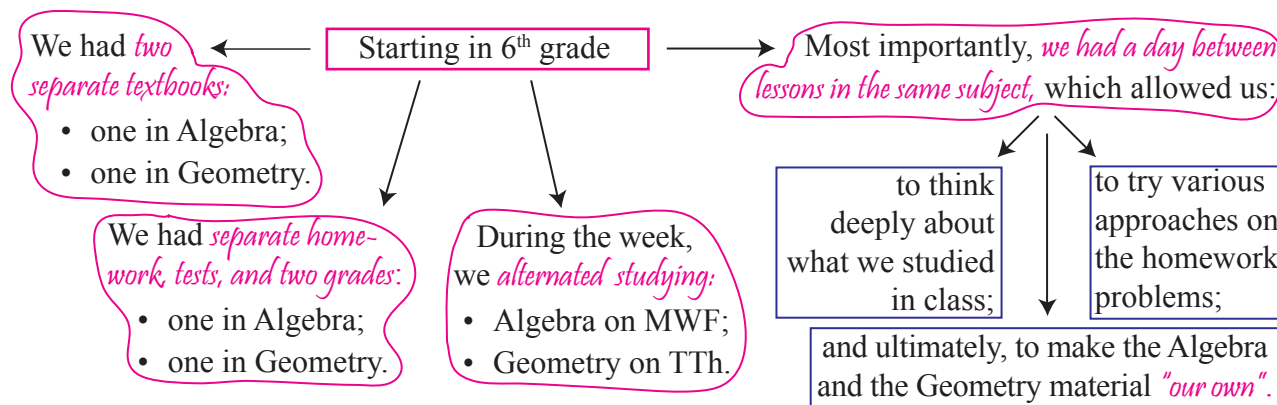
- Geometry has been sacrificed* in favor of
- Arithmetic in elementary school;
 - Algebra in middle school, and
 - Pre-Calculus/Calculus in high school.

The *Geometry Pillar will never be built* with the strength and height as it should be in order to hold up a robust and elegant Temple of Mathematics.

The lack of serious and consistent Geometry in early grades makes it impossible to overturn the “algebraic dominance” in the K-12 curriculum. Can this be changed? Can Geometry “Neverland” be replaced by *a healthy symbiosis between Geometry and the rest of mathematics*? Let us look at what could have been.

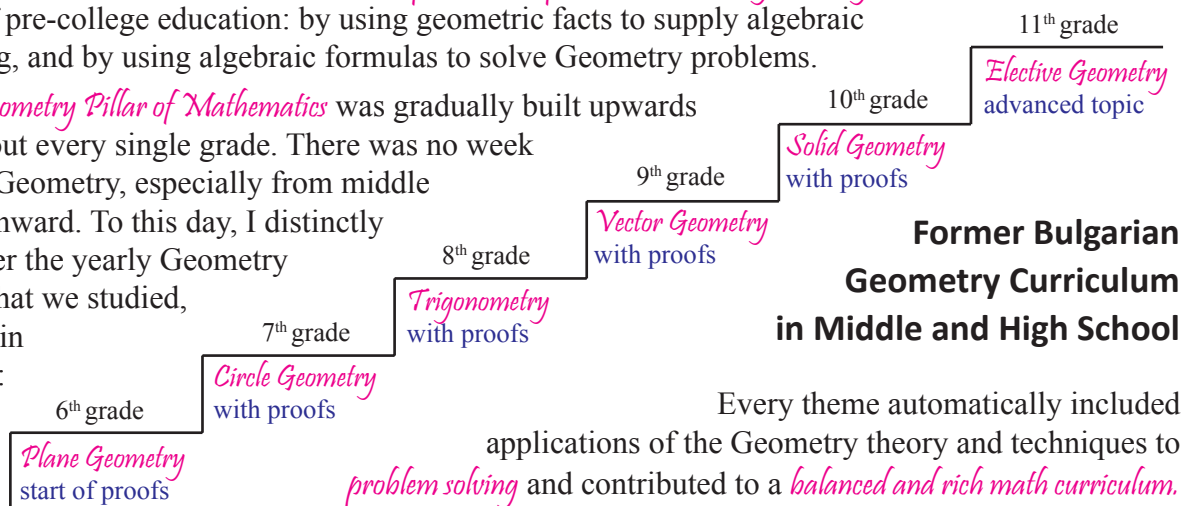
8.4. A Personal Lamentation Or ... the Golden Geometry Days of My Childhood

From my 25-years of experience teaching just about every undergraduate math course in college, the most mystifying formulas to my students, the least well understood, and (it goes without saying) the least frequently proven in pre-college years are the *trigonometric* ones. During the socialist times *in Bulgaria*, trigonometry was part of the standard 8th grade Geometry curriculum; the theory behind such formulas was systematically studied, and the formulas themselves were rigorously proven via geometric or, as needed, algebraic methods and applied to various problems for years in a row. In fact,



Although in two formally separate subjects, our studies of Algebra and Geometry were synchronized and mutually enriching. *A constant and deep relationship between Geometry and Algebra* was reinforced at all stages of pre-college education: by using geometric facts to supply algebraic reasoning, and by using algebraic formulas to solve Geometry problems.

The Geometry Pillar of Mathematics was gradually built upwards throughout every single grade. There was no week without Geometry, especially from middle school onward. To this day, I distinctly remember the yearly Geometry themes that we studied, starting in 6th grade:



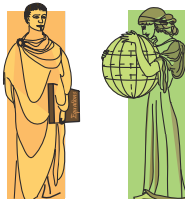
This is the *ideal symbiotic model of Geo-Algebra* that I would like for my children and grandchildren.

Unfortunately, high school math curriculum in the U.S. is driven, by and large, by college requirements and, let's face it, earning "bonus entrance points" to the more competitive universities. Hence, it is ultimately a *Calculus-driven curriculum*, regardless of whether students complete Calculus courses in high school, or not.

In turn, middle school math curriculum aligns itself with high school priorities. Backtracking, the pre-requisite to Calculus is Pre-Calculus, and the pre-requisite to Pre-Calculus is Algebra. Thus, Algebra "wins" by default, due to an established *Algebra-driven* and *Algebra-dominated environment*.

8.5. A Reality Check: Geo-Algebra in the U.S.?!

It is unrealistic (although not entirely impossible) to imagine that the pre-college curriculum in the U.S. can quickly change so much that it will embrace Geometry as an equal counterpart of Algebra, will split textbooks and classes into Algebra and Geometry ones, and will employ *the Geo-Algebra symbiotic model*.



The present textbook series capitalizes on a compromise between Eastern European and U.S. math educational systems and realities.

On the other hand, times **have** changed. *The Bulgarian curriculum has moved part-way towards the U.S. one*; e.g., newer Bulgarian textbooks feature both Algebra and Geometry, while preserving the mathematical integrity and pedagogical insights of a tremendously successful curriculum from the near past.

At a Glance: The New Middle School Curriculum Adapted to the U.S. Reality by Topic and Semester

At a Glance: The New Middle School Curriculum Adapted to the U.S. Reality by Topic and Semester			
6A Textbook		6B Textbook	Math 1 7A Textbook
Decimals operations, algebra with numbers; calculate efficiently; motion & word problems	Fractions, %'s operations; part of a whole; applications Exponents rules; properties Rational #'s operations with pos./neg. numbers; modulus; equations	Proportions properties, applications Polynomials variables, numerical values, degrees, operations, algebraic identities; factoring in degrees 2 and 3	Equations linear, factored quadratic, parametric; w/ absolute value; modeling, practical applications
Divisibility criteria for divisibility; CGD and LCM	Graphing, Data coordinates, diagrams/tables, arithmetic mean	Solid Geometry types of prisms and pyramids; surface area, volume; constructions, applications	Plane Geometry basic notions: pairs of angles, perpendicular/parallel lines; criteria, axioms, theorems
Plane Geometry altitudes; parallel lines; quadrilateral: perimeter, area; apply algebra	Plane Geometry circles, discs, polygons: perimeter, area; regular n-gons; constructions	Include: relevant formulas with variables; then substitute given numerical data.	Congruent Δ's criteria, isosceles Δ, perpendicular bisector, 30°-60°-90° Δ's
Solid Geometry rectangular parallelepiped: surface area, volume; applications	Recognize: expressions and equations. Split: solutions into "given," "to find," and calculations.	Apply: algebra tools to solve geometry problems.	Draw and Use: labeling of geometric objects. Explain in Words: what the variables stand for, important parts of solution, and the conclusion.
Write: correct, complete, and clear solutions.	Draw: pictures and diagrams		
			Math 2 7B Textbook
			Inequalities properties, one variable, linear; representations vs. equations; inequalities in Δ
			Square Roots Quadratic Eq'ns irrational numbers; algebraic expressions; quadratic f-la
			Congruent Δ's medians, angle bisectors, altitudes; constructions with a ruler and a compass
			Quadrilaterals parallelogram, trapezoid; theorems, proofs, constructions, applications
			Devise: algorithms for geometric constructions; Create: algebraic proofs.
			Math 3 8A Textbook
			Functions graphs; directly and inversely proportional; linear and absolute value functions; applications to equations and inequalities; graph of ax^2
			Transformations translations, rotations, reflections, central symmetries; congruences; definitions, properties; constructions, applications to polygons and circles
			Determine: the hypothesis and what to prove; Construct: a proof as a logical sequence with words and math symbols.
			Math 4 8B Textbook
			Systems of Equations and Inequalities linear w/ 1 or 2 unknowns, w/ modulus; modeling, applications
			Vectors operations; applications to midsegments and centroids
			Circle Geometry tangents; angles in a circle; geometric loci; circumcircles and incircles; famous pts in Δ; circumscribed and inscribed quadrilaterals
			Find: multiple ways to solve. Distinguish: between faulty and valid reasoning. Learn: from own experience and others' solutions.
			Math 5 8C Textbook
			Legend Algebra Geometry Statistics Number Theory Logic and Proofs

8.6. The Main Principles

The **Main Principles** in designing this middle school math curriculum are


**Continuity + Balance + Integrity
of the 4 basic building math blocks:**



1. **Algebra** is developed from an early stage, using numbers at first and then more and more frequently variables.
2. **Geometry** is developed alongside Algebra as an equal partner.



Algebra & Geometry support and reinforce each other via applications of each subject into the other.



3. **Logic, Proofs, and Problem Solving*** comprise the skeleton of the Math Temple, providing the “glue” that keeps together all parts of mathematics.

In the syllabus on p. XX, the Logic/Proof-Writing Topics are formally split into two semesters in each grade. In reality, they appear uniformly throughout the whole middle school curriculum.

4A. Number Theory, or the Queen of Mathematics,** makes a grand appearance through the topic of *Divisibility* in the 6A textbook (and later), to aid both logical development and practical algebra skills; e.g., finding efficiently *GCD* and *LCM*.

If you wonder what Number Theory is all about, try to decide if 9,876,543,210 is divisible by 15 *without actually dividing by 15!* The world of *whole* numbers is how Number Theory can be approximately described in middle school.



4B. Statistics and Applications are ubiquitous in the curriculum, aiming at data graphing, optimization, and approximations, and aided by physical experimentation.

* The role of Logic, Proofs, and Problem Solving is played by the Greek Titan *Atlas*, often pictured “holding the Earth on his shoulders”. In our situation, Atlas represents the *central Pillar of Mathematics*, which keeps the whole structure together.

** Number Theory is widely regarded as the Queen of Mathematics. Indeed, some of the most famous and hard math problems have been in Number Theory; e.g., Fermat’s Last Theorem. In our Math Temple, Number Theory is played by *Athena*, the Greek Goddess of Mathematics (also of wisdom, courage, inspiration, war strategy, arts, literature, and others). In the textbook series, Athena also supports the areas of Graphing and Data Analysis, or Statistics, and other applications.

9. Differentiate and Scaffold

Talking about differentiation in math classes is a “politically correct” and enticing buzzword to mention when advertising a curriculum and a school. Yet, by and large, this talk is not matched by the little done to truly support both ends of the spectrum... because it is hard to scaffold math lessons and because *it takes an enthusiastic math specialist to successfully cope with diversity in the math classroom.*

When mathematics is taught:

- in an organized and sound way,
- building a solid Math Temple for everyone, the number of students who “fall through the cracks” and lag behind is minimized. Students who are ahead, though, need something else: something beyond the regular curriculum that will still keep them topic-wise in the same math vicinity as the rest of their classmates. To this end, the exercises in the **Textbooks** are split into:

- 1 **Basic** (mandatory) problems;
- 2 **High Proficiency** (recommended) problems;
- 3 4 **Harder/Math Contest** (elective) problems.

The **Workbooks** provide solid “workouts” on the basic material. While everyone should complete a substantial portion of them, the Workbooks are an absolute must for the weaker students, who should work extensively with them.

In addition, a **Problem Collection for the Teacher** is developed for every grade to match the material studied in every chapter of the textbook. Some problems are grouped in multiple-choice tests, and some are open-ended and organized by topic. The material is gradated in three levels:

- **Level A:** basic material, building the fundamentals and providing practice for everyone.
- **Level B:** elective material, extending the knowledge for those who want to move forward.
- **Level C:** preparation for math contests for the more adventurous and/or advanced students.

9.1. Examples of Scaffolding

The 7A Textbook features the following exercises from the Chapters on Round Solids and Algebra on Polynomials that reach all parts of the difficulty spectrum:

1 (*Basic*). A right circular cone with a diameter of 12 cm has a slant height of 8 cm. Find the surface area of the cone.

2 (*High Proficiency*). We have cut out a cone from a cylinder with volume 72 cm^3 , as shown in the drawing. If the radius and the height of the cone are correspondingly $1/2$ of the radius and $1/2$ of the height of the cylinder, find the volume of the cone.

3 (*Hard, but doable w/ school material*). Show that the difference of squares of any odd number and the number 1 is divisible by 8.

4 (*More advanced material/ideas*). Prove that the sum of the cubes of two numbers that are 2 apart is divisible by the double of the number between them.

Occasionally, the textbook uses the eye-icon to signify a fact or technique that is *beyond the textbook material*. For example, Lessons 69 and 71 on Factoring of Polynomials challenge the reader:



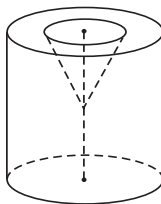
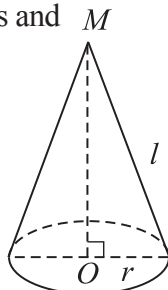
There is a method to determine whether a quadratic trinomial is *reducible* (can be factored) or is *irreducible*. This method is known as “*completing the square*”; e.g.,

$$\begin{aligned} x^2 + 6x - 7 &= x^2 + 2 \cdot x \cdot 3 + 3^2 - 3^2 - 7 = \\ &= (x + 3)^2 - 16 = \\ &= \boxed{(x + 3)^2 - 4^2} = \\ &= (x + 3 + 4)(x + 3 - 4) = \\ &= (x + 7)(x - 1) \end{aligned}$$



Factor the polynomials:

- $x^4 + 4$;
- $x^4 + x^2 + 1$;
- $a^4 + a^2b^2 + b^4$.



9.2. What is Math 5/8C Textbook?

This U.S.-adapted program starts in 6th grade with the 5th grade Bulgarian material; that is,

- In the beginning, it lags a *full year* behind its Bulgarian counterpart.
- By the end of 6th grade, it lags only *half a year*.
- By the end of 8th grade, the difference is cut down to only a *quarter of a year*.

All of the original 5th-8th grade Algebra material and most of the original Geometry material is covered through the current 6th-8th grade textbooks.

The harder half of the original 8th Geometry material on *Vectors and Circle Geometry* comprises the extra 8C textbook. It is intended *for the more advanced students*, perhaps, to be covered as:

- an elective 8th grade course in Geometry.

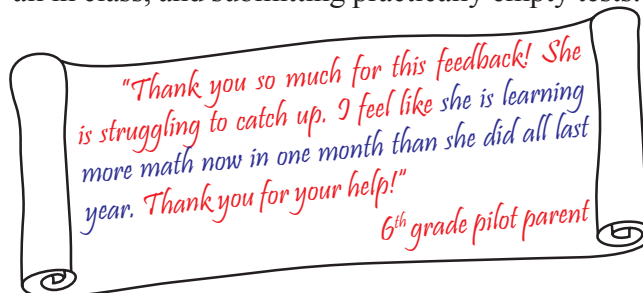
The extra material in the 8C textbook is considerably ahead of the standard Geometry material in U.S. high schools and it will be:

- a *superb preparation* for those who would like *to be challenged and to excel in Geometry*.

9.3. Does the Scaffolding Work?

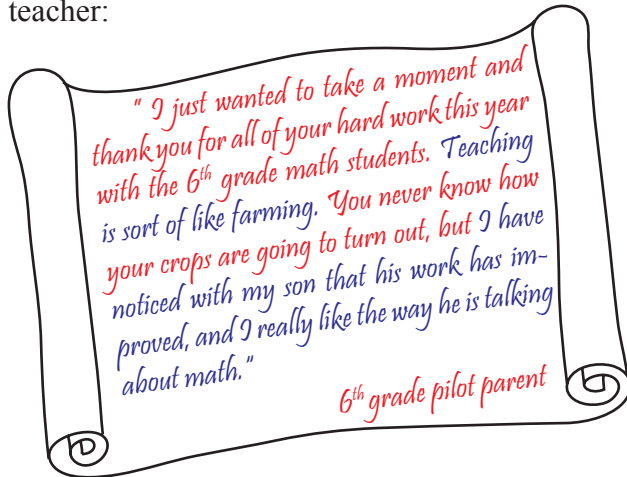
Does the program really benefit all levels of students? Here is some anecdotal evidence.

Answer 1: A month after the start of the pilot 6th grade program, the teacher received a letter from a parent of a student who was struggling tremendously in the beginning, not participating at all in class, and submitting practically empty tests:



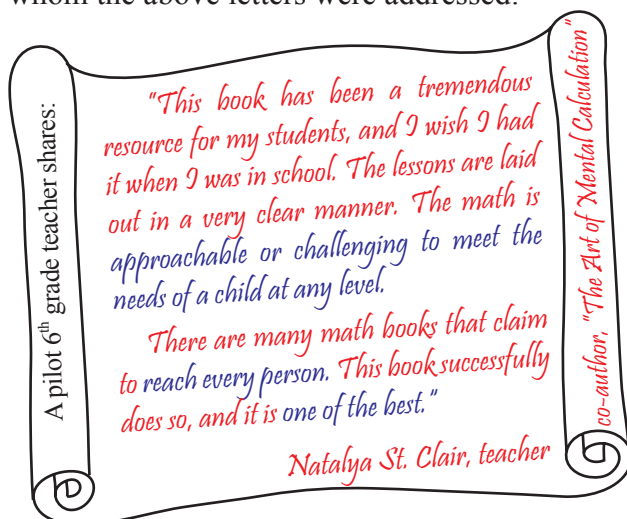
The student gradually progressed to very good performance both on tests and in class, showing enthusiasm and understanding of material to which she had not been even exposed before but was now fired up to learn.

Answer 2: Several months after the start of the pilot program, a parent of a student who had always excelled in math wrote to the 6th grade pilot teacher:



The new program has worked exceptionally well on a number of students who had earlier been identified as having severe problems with learning mathematics. These students have gradually improved and excelled with the new curriculum, and questions about their future success with mathematics have become obsolete. Incidentally, a few of the girls improved so rapidly that 2 months into the program they had to be moved to the more advanced 6th-grade pilot class.

Answer 3: Here is the truth as seen through the eyes of the very first pilot teacher in the U.S. who worked with this program in 6th grade and achieved what was impossible before, and to whom the above letters were addressed:



10. The Value of a Teacher

As with any textbooks, the present textbooks are just that: *a basis for learning*. They will come alive in the classroom *only through the teacher*.

Let us not deceive ourselves: there is no universal panacea for the middle school math problems in the U.S. And although the present textbooks have been hailed by pilot parents as:

- the best textbooks they have ever seen;
- the most suitable for their children;
- a resource to easily follow and learn from at home if a child is absent from school or did not understand something in class...

...still, *a mediocre teacher will do a mediocre job even with the best textbook in hand*.

The fact that many middle school math positions are filled by non-specialists in math is alarming and is one of the major reasons for the current general downfall of U.S. middle school math. A non-specialist does not have an “eagle view” of the full middle and high school math curriculum. Although, locally, he/she might be able to teach well some of the material, a lot of damage is done by a teacher’s limited knowledge and *lack of understanding of what is important in the long run, even if hard to teach*.

The situation is similar to dropping by parachutes the whole math class in the middle of a jungle, giving each student a flashlight, and waiting outside the jungle to see who will come out alive. A better chance of “survival” will be ensured if the students are *given a map of the area, taught how to read and use it, and equipped with the necessary tools, knowledge, and skills* to efficiently cope with the difficulties that will inevitably occur on the way out of the jungle. And *the teacher has to be there, with the students, every step of the way,...* not just waiting for them outside of the “danger zone”. I personally never thought of school math in this “jungle vision” until my own children experienced several different U.S. math programs and the image of the “jungle” resembled more and more powerfully the reality of pre-college math education.

More often than not, non-specialist math teachers in middle and elementary school, for a variety of reasons, decide to change the order, depth, and emphasis of the math program they are teaching from. Not knowing the “big picture” of how mathematics will develop over high school and then college years (and perhaps, not caring about this big picture, since it is not part of their day-to-day job duties), their actions unintentionally destroy the logical structure and goals of the program, whether it is a good program or not. (See earlier discussion of this situation in Section 5: The Temple of Mathematics.)

Fortunately, the new math program:

Is *straightforward to follow*: every lesson is on 2 pages only!

When each lesson in the textbook is on 2 pages only, with 6-to-10 exercises, and when the concepts are practiced in the next lessons and incorporated in new situations for better understanding, it is *very hard to justify staying on one particular lesson for a week or so* (and killing everyone’s enthusiasm in the process).

(Most importantly!)
Ensures accountability from the teachers.

The program is so crystal clear that *anyone can track where the students are and what they have covered* at any time during the school year. By “anyone” we mean parents, administrators, and random visitors alike. It is essential, though, that a copy of *the textbook resides at every student’s home*; there is nothing to hide in the program, while there is so much to learn from reading the textbook at home!

Does not require any search for extra materials elsewhere: the textbook, workbook, and problem collection *contain everything for a successful completion of the program.*

Due to the *natural transparency of the material*, it is impossible to hide behind phrases like “too much,” “too hard,” or “not well organized” “to be covered all in one year”. And the *program does depend on the steady and full completion of the material* in every middle school year, in order for the next year to be successful.

A Good Choice for a Teacher
for this curriculum
would take into account:

The Mathematical Level

A teacher with a *solid math background* who can *see several steps beyond middle school*, e.g., the high school curriculum, and can recognize:

- the math concepts,
- types of solutions, and
- features of analytical thinking

that are indispensable and must be taught and reinforced in middle school, despite how hard it might be to teach them to a young audience unaware of them.

The Pedagogical Level

A teacher who can *quickly adapt to new situations* and turn them into an educational advantage for the students.

A teacher who is constantly attuned to the weaknesses and strengths in students’ background and skills, and who emphasizes accordingly features of the curriculum in a balanced way to match and improve the classroom situation.

A teacher who *can communicate mathematics to young students*.

The Level of Dedication

A teacher who is *dedicated to the well-being of the students* and who will sacrifice his/her own desire to “shine on the career ladder” to providing a robust math education to his/her young charges.

A teacher *who believes in and is ready to undertake and follow closely the new curriculum*; who is ready to face the mathematical and cultural difficulties that will inevitably arise in introducing such a program.

A teacher who *will persevere*.

11. Can a U.S. Middle School be Successful with this New Math Curriculum?

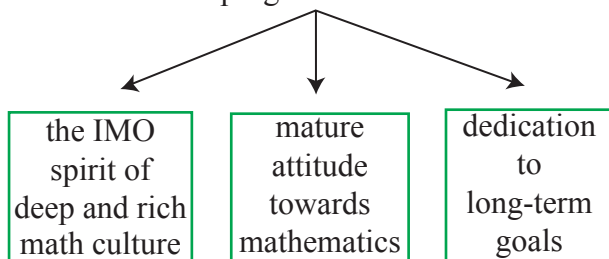
What kind of question is this?! Let us analyze the situation as a mathematician would, putting aside our own ambitions and de-politicizing math education.

11.1. The U.S. Gene Pool

Of course, U.S. students can do on the average just as well as (if not better than) their Bulgarian peers, and, for that matter, any peers around the world. As far as the “gene pool” of the U.S. is concerned, it is probably the best mixture in the world.

The cream of the crop of the U.S. students have performed at the top places and occasionally have made the number 1 team in the world (!) at the International Mathematical Olympiads (IMO)*, the preeminent world math competition for pre-college students that tests **not** speed but *depth and originality of mathematical thinking*.

The new math program emulates:



It is true that very few students will end up being math Olympians, and we will not give false hopes that the current program is training students for the IMOs. However, the expectation is that:

Everyone who is touched by this program will build a solid Temple of Mathematics, where later, if desired, will hang the “fancy chandeliers” and “brilliant art pieces”. More importantly, the personal connection with mathematics will remain all through life to be enjoyed and cherished.

11.2. Can girls succeed in math?

Upon coming to the U.S., I was shocked by this question. I was raised in a provincial town in Bulgaria and made it as a high school student to two IMOs, winning silver medals. There was another girl on the Bulgarian team for the 2 years I was there.

It took the U.S. 25 years of participating at the IMOs before the first U.S. girl, Melanie Wood, qualified for the IMOs. I was privileged to train the U.S. national team when the Melanie competed at the IMOs, also earning two silver medals. Later, I participated in the training of the other two U.S. girls, who went on to win gold medals at the IMOs.

To cut the long story short: anything is possible and *gender does not matter as far as mathematical talent and success are concerned*.



Gender apparently matters in the social and cultural aspects of education, and this severely handicaps many U.S. girls who might have otherwise become excellent mathematicians.



Decades after competing at the IMOs, I learned that Bulgaria is the top country in the world in sending the largest number of girls**, 21, to the IMOs. Germany and Russia are next, with 19 and 15 girls, respectively. USA has only 3 girls so far.

Perhaps, *it is time for the U.S. educational system to follow examples of other programs from around the world* that have been hugely successful in raising gen-

* The IMO takes place every summer in a different country. Each country's team has 6 students. As opposed to standardized tests, there are only 6 problems over 2 days, 4.5 hours each day. The solutions are written as rigorous mathematical proofs.

** See “Cross-Cultural Analysis of Students with Exceptional Talent in Mathematical Problem Solving,” by Titu Andreescu, Joseph A. Gallian, and Jonathan M. Kane, <http://www.ams.org/notices/200810/fea-gallian.pdf>.

erations of women who are highly educated in math and other STEM (Science, Technology, Engineering and Mathematics) disciplines, and who have been brought up to think of men and women as intellectually equal.

11.3. The Needed Cultural Shift

The current middle school math program originating in Bulgaria necessitates a cultural shift in a U.S. school community so that the program takes roots and yields results. For starters, the school must make:

- *math a top priority in its academic curriculum*

and follow unwaveringly in its determination to overturn the middle school math situation to the huge benefit of its students.

11.4. Continuity of the Curriculum

It is most important and advantageous for everyone that:

The **same** teacher continues with the **same** students throughout middle school.

The alternative has struck me as a very bad way to organize middle school math education. Indeed, when a person teaches the same grade year after year he/she:

- gains no understanding of the “big picture” of pre-college math education;
- shares no responsibility for the continuation of the program in the next grade (and a lot of time will be wasted in the beginning of each school year for the new teacher to test and get attuned to the new students);
- has no incentive (or responsibility) for reaching the final goals of the program.

Yes, I am acutely aware of the counterargument:

“I do not want a bad math teacher to teach my kid for all of middle school. I would rather bite the bullet of a bad math teacher one year, so as to get the good math teacher the other year.”

In summary, many parents are (unwittingly) willing for their kids to endure on the average a mediocre middle school education. Can you imagine what kind of damage the “bad” teacher would do to the Temple of Mathematics during that, hopefully, single year of “bad” math lessons? Would the next, “good” teacher have the opportunity, or even the incentive, to rebuild the Temple AND cover the new material on top of that? Probably not.

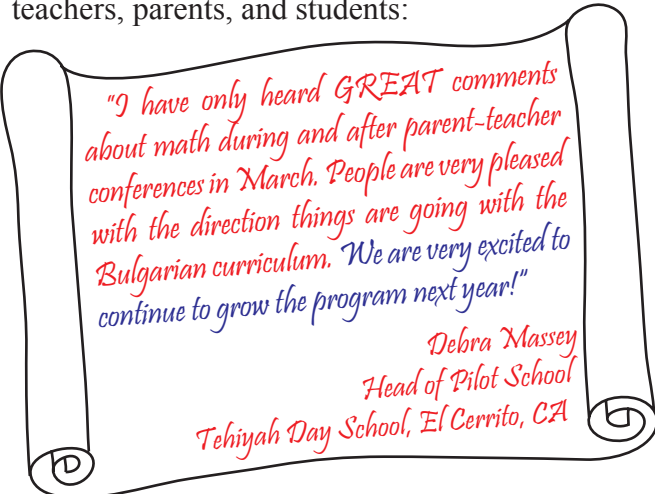
This kind of “averaging math teaching” is another major reason for the tremendous difficulties in U.S. middle school math education. *A teacher must be the asset of math education*; equally important (if not more important) than the actual math curriculum! And so, the truth must be spoken:

- a bad math teacher must be let go, to prevent more irreversible damage to our kids;
- a good math teacher must be treasured and given opportunities to further develop.

11.5. Local Support

There is no easy remedy and no miracle solutions to math educational problems anywhere around the world. The new math Program will need all the support it can get from parents and students, in addition to teachers and administration. The Program will depend on the school community’s belief in it and commitment to it.

Approaching the end of the first pilot year, here is what the Head of School shared, after extensive observations and conversations with pilot teachers, parents, and students:



“I have only heard GREAT comments about math during and after parent-teacher conferences in March. People are very pleased with the direction things are going with the Bulgarian curriculum. We are very excited to continue to grow the program next year!”

*Debra Massey
Head of Pilot School
Tehiyah Day School, El Cerrito, CA*

12. Within the Global U.S. Picture

At a sectional meeting of the Mathematical Association of America (MAA) in 2014, I heard a captivating presentation by *Dr. Phil Daro*, one of the three principal writers of *the Common Core Mathematics Standards* (CCSS-M). The talk addressed the biggest education problem in school mathematics, on which the CCSS-M writing team focused:

"The mile wide, inch deep" configuration of the American school curriculum.

As a consequence, Dr. Daro would ask any teacher:

"Are you teaching

Mathematics or *Answer-Getting Strategies?*"

This question fully resonated with the educational philosophy with which I was brought up back in Bulgaria. Dr. Daro explained that the core change incorporated in the CCSS-M is:

A shift in the perspective toward mathematics taught in schools:

Less emphasis on a fragmented multitude of special solution methods, especially tricks that avoid underlying mathematical principles.

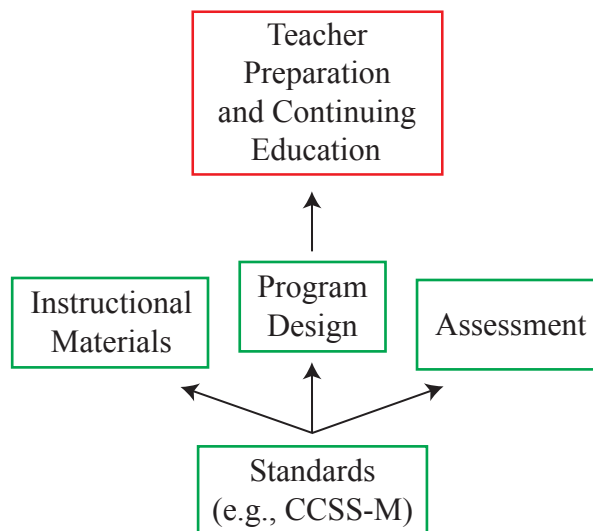
More emphasis on core principles and practices that unify and focus mathematical thinking.

Dr. Daro assembled the major pieces of the gigantic math education picture in the U.S.:

"Standards can help solve this kind of curriculum architecture problem, but we all face more detailed and difficult challenges of instructional materials, assessment, and program design, as well as the even more [difficult] area of teacher preparation and continuing education."

*Dr. Phil Daro
A Principal Writer
Common Core Mathematics Standards*

Visually, the scheme below represents the levels of change in the U.S. math education that are envisioned by the CCSS-M writing team:



At the time, I had no idea that I would soon launch the translation, adaptation, and implementation effort of the present curriculum. Inspired by Dr. Daro's talk, I invited him to meet in person the then-leaders of Tehiyah Day School. Perhaps, not entirely by serendipity, this meeting marked the path for the new math curriculum: as it turned out, the new Bulgarian curriculum matched very closely the CCSS-M.

During the 2016-2017 academic school year, I spent a day every week working with the pilot classes, training the pilot teachers, and learning myself how to adapt the curriculum to the U.S. reality. The instructional materials, program design, and assessment of the new adapted math curriculum for 6th grade are ready, and the adaptations of the 7th and 8th curricula are in preparation.

Coming full circle, two years after his memorable MAA presentation Dr. Daro reviewed the U.S.-adapted version of our new math program and assessed it as a

"Middle and High School Math Program that is deeply aligned with the Common Core Mathematics Standards".

13. Math 1 (7A) Program by Topics and Lessons

Algebra in Numbers and Symbols

Ratio. Proportion

- Definitions, Independence of Units L36
- Mean and Extreme Terms L36
- Proportionality, Direct/Inverse L37
- Variation, Constant of Variation L37
- Basic Property, Fourth Proportional L38
- Algebraic Properties L39
- Extended Ratio L41



Integer Expressions. Polynomials

- Rational Expressions L46
- Variables (Independent/Dependent) L46
- Constants, Parameters L46
- Integer Expressions/Polynomials L47
- Allowable Values (AV, Domain) L47
- Equal Expressions L47-48,59,70

Monomials:

- Definition, Standard Form, Coefficient, Degree L48
- Similar L49-50
- Adding/Subtracting L49-50
- Multiplying, Powers, Dividing L51

Polynomials:

- Definition, Standard Form L52
- Bi-, Tri-, Quadri-nomials L52
- Coefficient, Degree L52
- Adding/Subtracting L53
- Multiplying with a Monomial L54
- Multiplying with a Polynomial L55-56

Algebra with Polynomials

- Equivalent Expressions, Identities L47,59,70

Quick Multiplication of Polynomials:

- Degree 2 Formulas L60-64*,75
- Degree 3 Formulas L73-75

Factoring Polynomials Using:

- Common Factors in Monomials L65
- Formulas for Quick Multiplication L66,76-77
- Grouping L67
- Combining Methods L68-71,76-77
- Equivalence Transformations L48,70

* L60-64 = Lessons 60 through 64

Spatial Geometry

Polyhedral Solids

Prism. Right and Regular Prism:

- Definitions, Drawing, Elements L8-9
- Net, Surface Area (Lateral, Total) L10-11
- Volume L12; Mixed Exercises L13

Pyramid. Right and Regular Pyramid:

- Definitions, Drawing, Elements L14-15
- Net, Surface Area (Lateral, Total) L16-18
- Volume L19; Mixed Exercises L20

Round Solids

Cylinder. Right Circular Cylinder:

- Definitions, Drawing, Elements L23
- Net, Surface Area (Lateral, Total) L24
- Volume L25

Cone. Right Circular Cone:

- Definitions, Drawing, Net, Elements, L26
- Surface Area (Lateral, Total) L27-28
- Volume L29

Sphere. Ball:

- Definitions, Drawing, Elements L30
- Surface Area L30,32
- Volume of a Ball L31-33



Geo-Algebra

Algebra in 3D Geometry:

- Lateral/Total Surface Area L9-11,13,15-17,20,24,27-28,30,32
- Volume L12-13,19-20,25,29,31-33
- Prime Factors: Surface/Volume of a Sphere L30,32

Geometry in Algebra:

- Angles w/ Extended Ratios/Diagrams L42-43
- Proportions: Maps, Scales, Distances L40
- Extended Ratios L41,44
- Polynomials in Areas L63
- $(a + b)^2 = a^2 + 2ab + b^2$

Geometric
Proof
L60



Applications

Physical Experimentation:

- Nets of Solids L10,16,18,23-24,26
- Revolve about an Axis L23,26,28-31
- Volumes L12,19,25,29,31,33-34

Proportions and Extended Ratios:

- Maps, Scales, Distances L40
- Data/Statistics/Communications L42-43
- Business, Cooking, Geometry L41,44

Optimization:

- Max/Min of Polynomials L53,56,63,69,71-72,75

Approximation:

- Powers a^2 , a^3 for $a \approx 1$ L70,75

Number Theory:

- Divisibility of Polynomials L65,70-71,75,77

Legend

Algebra

Geometry

Applications

Geo-Algebra

Logic and Proofs

Logic

See Relations:

- Types of fractions, operations, conversions L1
- Parts, percents, types of equations L1
- Rational numbers, operations, properties L2
- Powers, bases, exponents, properties L2
- Formulas for circles and polygons L3
- Btw surface/volume formulas L10-31
- Btw variable quantities in problems L37,38

Recognize and Distinguish:

- Statement vs. its negation? L2
- Regular vs. non-regular prism/pyramid? L8,14
- Right vs. oblique solid? L8,23,26
- Ratio vs. proportion? L36
- Direct vs. inverse variation? L37
- Rational vs. integer expression? L46-47
- Unknown, variable, parameter, or constant? L46
- Independent vs. dependent variable? L46
- Equal** vs. equivalent expression? L47,59
- Equality vs. identity? L59
- Reducible vs. irreducible? L65-72,74-77
- Degree 2 vs. degree 3 formulas? L59-77
- Formal vs. informal definition? L8-77

Problem Solving

Calculate Efficiently:

- Expressions; polynomials L5,24,50-55,61,75-77

Develop 3D Vision and Skills:

- Learn to draw solids in 3D L9,15,23,26,30-31
- Construct nets of solids in 2D L10,16,18,27
- Revolve parts to get the whole L23,26,28-31

Read and Understand:

- When approximations are expected L70,75
- When to avoid approximations L24
- Alg. formulas in words/conceptually L60,62,73
- Alg. formulas via a geometric idea L60

Read, Interpret, and Solve Word Problems With:

- Maps, scales, distances; geometry L40-41,44
- Data processing, graphing L42-43

Writing Solutions and Proofs

Recognize Impossibilities and Possibilities:

- Unrealistic values of a variable L47
- Some methods cannot prove identities L59
- Irreducible/impossible to factor now L64,69
- A failed idea doesn't mean unsolvable L67

Write:

- Correctly, completely, clearly L5
- In correct geom./alg. notation L8-35, 36-77

Build a Network:

- Elements and types of solids L8,14,23,26,30-31
- Vocabulary of ratios and proportions L36
- Vocabulary of variations L37
- Properties of proportions L38-39,44
- Vocabulary of expressions L46-47
- Vocabulary/operations with monomials L48-51
- Vocabulary/operations with polynomials L52-57
- Formulas to factor/multiply polynomials L59-77

Organize Data and Possibilities:

- On an axis/coordinate system/grid L2-4,8-35
- In a table/diagram: ubiquitous throughout the text
- Learn to multiply polynomials in stages L54-55
- Split into cases L2,59,69

Reduce to a Previous Problem:

- 3D to 2D geometry L8-11,13,16,23-24,27,30
- Complex to simpler solids L12,19,25,29,31
- Proportions to simpler proportions L39
- Operations on monomials to numbers L49
- Operations on polynomials to monomials L53-55

Make Connections in Solutions:

- Recognize unnecessary information L17
- Factor to find radius of a sphere/ball L30-34
- Add parts to get the whole L32-34
- Subtract from the whole L33-34
- Combine similar monomials L49-50
- Turn a difference into a sum L49,53-54,61
- Substitute an expression by a variable L64

Search for Multiple Solutions in Working with:

- Identities and non-identities L59
- Algebraic formulas L61,64

Select and Apply Different Methods to:

- Factor polynomials L68-69
- Prove identities/multiply polynomials L70

Split Geometric Problems Into:

- Picture, Given, Prove/Find, Solution L8-35

"Proofs" via Physical Experimentation:

- Volumes L12,19,25,29,31

Formula(s) for Surface Area of:

- a sphere *was introduced without proof* in L30
- a prism, pyramid, cylinder, and cone *were proved* in L10,16,24,27

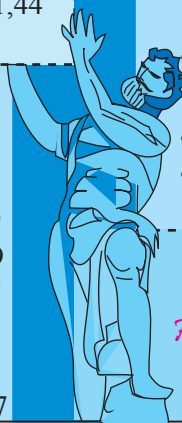
Study and Review

Review: Of 6th Grade Material L1-5

How to Study: For Tests L7

Summarize: Topics L21,34,44,57,71,77

Practice: Tests L6,22,35,45,58,72



** "Equal" expressions were discussed in L47, but we formally called them "equivalent" only later in L59.

14. How to Work w/ Math 1 (7A) Materials

The **Math 1/7A Textbook** is for use in class as the main source of material to be covered:

- *The Textbook Examples* are the core of the class discussion and work.
- *The Textbook Exercises* can be done in class or assigned for homework.
- *Answers to all Exercises* are in the back of the textbook; hence, writing only answers will bring no credit, while good reasoning and organized calculations will earn a lot of credit.

All Lesson Exercises in the **Math 1/7A Workbook** (marked as L__) are identical to the examples in the textbook, with spaces for writing solutions. The Lesson Exercises are intended for classwork.

The Worksheets in the **Math1/7A Workbook** (marked as W__) thematically match the textbook, lesson-by-lesson. The Worksheets are intended mostly for independent work in class or homework. They are usually (but not always!) geared towards students who need more help and extra support with the material.

The **Math 1/7A Problem Collection for the Teacher** can be used for tests and supplementary work with students who excel with the material.

The **Math 1/7A Pedagogical Guide** will reveal to the teacher ideas and strategies gained by decades of teaching in the U.S. and by the practical wisdom and experience of the Program's pilot teachers.

15. Credits

“The Mathematical Association of America’s *American Mathematics Competitions* program (AMC) leads the nation in strengthening the mathematical capabilities of the next generation of problem-solvers. Through classroom resources and friendly competition, the MAA AMC program helps America’s educators identify talent and foster a love of mathematics. The MAA AMC program positively impacts the analytical skills needed for future careers in an innovative society.

The American Mathematics Competitions are a series of examinations and curriculum materials that build problem-solving skills and mathematical knowledge in middle and high school students.”

(From <http://www.maa.org/math-competitions>)

Every year, from hundreds of thousands of high school and middle school students the AMC selects a team of six U.S. students and trains them to represent the USA at the International Mathematical Olympiad (IMO). The sequence of national tests that leads to the final 6 team members consists of AMC10/12, AIME (American Invitational Mathematics Exam, USA(J)MO (USA (Junior) Mathematical Olympiad), and a series of qualifying tests during the summer training program before the IMO.

AMC8 is a preparation round for the younger students, in grades 8 and below. It does not count as a qualifying round, but it has similar features and structure as the AMC10/12 round: 25 multiple-choice problems for 40 minutes.

The following 16 problems were selected from AMC8 and used with permission of the MAA in the Math 1/7A Textbook (the same problems were used in the 6B Textbook):

- p. 9: Problems 1 and 22 from 2007;
Problem 9 from 2004;
- p. 3: Problem 12 from 1999;
- p. 3: Problem 5 from 2003;
Problem 22 from 1999;
Problem 7 from 2004;
- p. 13: Problems 20 and 24 from 2004;
Problem 4 and 16 from 1999;
Problem 4 from 2001;
Problems 22 and 25 from 2003;
Problems 9 and 19 from 2000.

16. Math Taught the Right Way

In addition to running the Tehiyah Day School math program on this textbook series for a second year, we opened an weekly extension program of the Berkeley Math Circle at UC Berkeley called “*Math Taught the Right Way*” (MTRW).

MTRW in **fall 2016 and spring 2017** ran on Thursdays, with a total of 70 students coming from 24 schools and 15 cities in the eastern part of the San Francisco Bay Area. The three program groups were selected mainly by age. As in the spirit of the Berkeley Math Circle, younger, more advanced students were promoted to higher groups. The fall groups were named 6A, 7A, and 8A, and their spring successors 6B, 7B, and 8B.

6A/6B Group:

Taught by:

- *6A/6B Algebra (5-6pm)* Elysee W.-Egolf
- *6A Geometry (6-7pm)* Natalia St. Clair
- *6B Geometry (6-7pm)* Ellen Kulinsky
- *6A/6B Problem Solving (7-8pm)* Elysee W.-Egolf

7A/7B Group:

- *7A/7B Problem Solving (5-6pm)* Petra Munih
- *7A/7B Geometry (6-7pm)* Norm Prokup
- *7A/7B Algebra (7-8pm)* Kelli Talaska

8A/8B Group:

- *8A/8B Proofs (6-7pm)* Zvezdelina Stankova
- *8A/8B Geometry (7-8pm)* Norm Prokup

The 12 classes in pink were taught exclusively from the current adapted curriculum. In the 15 weekly 3-hour session during the fall,

- *6A Algebra* covered Chapters 1-2 on *New Basic Concepts* and *Fractions/Decimals*. *6A Geometry* completed Chapter 3 on *Geometric Figures in the Plane* and half of Chapter 5 on *Rectangular Parallelepipeds*. *6A Problem Solving* studied Chapter 4 on *Divisibility*.
- *7A Algebra* covered Chapters 4-5 on *Proportions and Integer Expressions* in 7A curriculum. *7A Geometry* worked on Chapters 2-3 on *Polyhedral and Round Solids*.
- *8A Geometry* covered the *Vector Geometry* chapter in 8C curriculum.

In the 15 weekly 3-hour sessions in spring 2017,

- *6B Algebra class* covered Chapters 6, 8, 9 on *Common Fractions*, *Exponentiation*, and *Rational Numbers*. *6B Geometry* finished Chapter 5 on *Rectangular Parallelepipeds*, and completed Chapter 10 on *Circles and Polygons* and Chapter 11 on *Graphing and Data*. *6B Problem Solving* concentrated on Chapter 7 on *Applications of Fractions* to word problems.
- *7B Algebra class* covered Chapters 6 on *Algebra of Polynomials* and half of Chapter 8 on *Equations* in 7A-7B curriculum. *7B Geometry* completed Chapter 1 on *Basic Plane Figures* and part of Chapter 3 on *Congruent Triangles*.
- *8B Geometry* covered the *Isometries* chapter from 8A curriculum and the first half of the *Circle Geometry* chapter in 8B curriculum.

The four classes in blue were based on chapters of “*A Decade of the Berkeley Math Circle - the American Experience*,” edited by myself and Tom Rike, and exposed students to math circle material:

- *7A Problem Solving: Combinatorics* (vol. I);
- *7B Problem Solving: Rubic’s Cube, Part I* (vol. I) and *Introduction to Group Theory* (vol. II);
- *8A Proofs: Proofs, Part I and Induction* (vol. I);
- *8B Proofs: Stomp. Games with Invariants* (vol. I), and *Monovariants, Part II* (vol. II).

The **Berkeley Math Circle Summer Program 2017** can be partially viewed as a third part of the **MTRW Program**, to complete the current 6th-8th grade curriculum plus some math circle topics. The faculty who will teach at the program are:

- Dmitry Fuchs, Ph.D., UC Davis
- Elysee W.-Egolf, B.A. UCB, Bentley High
- Evan O’Dorney, BMC alumnus, Princeton
- Hai-Main Luu, BMC asst, UC Berkeley
- Kelli Talaska, Ph.D., UC Berkeley
- Laura Pierson, BMC alumna, Harvard
- Norm Prokup, Ph.D., College Prep, Oakland
- Petra Munih, Ph.D. Biophysics/Structural Biology
- Pratima Kapre, Ph.D., Santa Clara University
- Quan Lam, Ph.D., MBA, Office of UC President
- Sam Vandervelde, Ph.D., Proof School
- Zvezdelina Stankova, Ph.D., UC Berkeley

17. Acknowledgments

The adaptation to the U.S. reality of the language and math notation in this middle and high school math program, as well as the technical and mathematical correctness of the series, is due to the dedication and selflessness of *Tom Rike*, my long-time collaborator and co-editor. Tom, you are *amazing* (and that's not hyperbole)!

Natalya St. Clair was the very first pilot teacher, who moved from across the country to bravely undertake the new math Program in fall 2015 with the two 6th grade classes at Tehiyah Day School. Her enthusiasm for mathematics was evident each and every day in her lively lessons, sharing with her students her joy in teaching math. She also had numerous suggestions for adaptation, based on her daily experience with the Program on the U.S. turf, and offered constructive criticism of the revised version of the series. *George Khasin* and *Laurie Gold* continued the pilot Program in spring 2016 with a steady rhythm and care for the students, and *Andrea Jedwab* piloted the 7th grade class in fall 2016/spring 2017. *Kelli Talaska*, who teaches at the MTRW Program, also provided invaluable editorial comments on the textbook series.

Going further back, *Elise Prowse*, the former Interim Head of the pilot school and a math educator herself, recognized the power and depth of the new Program (while it was still in the original Bulgarian language!) and laid down the path of its implementation in practice. *Debra Massey*, the present Head of School and a parent herself, strongly backed the Program and supported it all the way through the crucial first pilot year.

The middle school math Program at Tehiyah Day School would not have had a stable basis to land on, had it not been for the Singapore Math Program in grades K-5, introduced in 2003 by *Professor Alexander Givental* of UC Berkeley and his wife, *Laura Givental*, the leader of the Berkeley Math Circle-Elementary Division for grades 1-4. The school can and will improve the efficiency of implementing the Singapore Math Curriculum at the elementary school grade levels. Still, the mere fact that Singapore Math precedes the pilot Math Program already immensely increased the chances of success of this pioneering Project.

It goes without saying that the vision of *Professor Georgi Paskalev* and his collaborators, *Zdravka Paskaleva* and *Maya Alashka*, is providing the generating power for this translation and adaptation effort. The success of the original Bulgarian textbook series is no surprise. The novel ideas of overall design, the simplicity and elegance in organization, and the balance between depth and breath of the material, are unsurpassed.

Any textbook series needs a Publisher. And what more suitable Publisher of the new textbook series is there than the original “Archimedes” Publishing House! Its Director, *Professor Plamen Paskalev*, undertook the Project with no reservations, investing in a future based on the common ground of two very different math educational systems: the U.S. and the Bulgarian systems.

Angelina Avramova, the graphic designer of the original series and the technical editor of the new series, prepared the materials for publication with efficiency and professionalism that would put any anxious collaborator at ease and that would make any publisher proud.

Looking forward, *Dr. Phil Daro*, one of the three Principal Writers of the Common Core Standards, and *Dr. Oleg Gleizer*, the Academic Dean of Geffen Academy at UCLA and Chief Curriculum Developer and Lead Instructor at the Los Angeles Math Circle, reviewed and endorsed the adapted Program. *Dr. Oleg Gleizer* and *Dr. Sibyll Catalan*, the Head of Geffen Academy, are working towards adopting the Program there. *Dr. Doug Ensley*, the Deputy Executive Director of the American Math Competitions made available about a dozen AMC8 problems for inclusion in the current 7B Textbook.

The daily work of the pilot 6th and 7th grade students and *the constant support of their parents* at Tehiyah moved this program from paper to real life and gave it meaning and purpose. The translation and adaptation effort would have been inconceivable without the understanding and love of my family: my husband, *Dima*, and our children, *Lily* and *Oleg*.

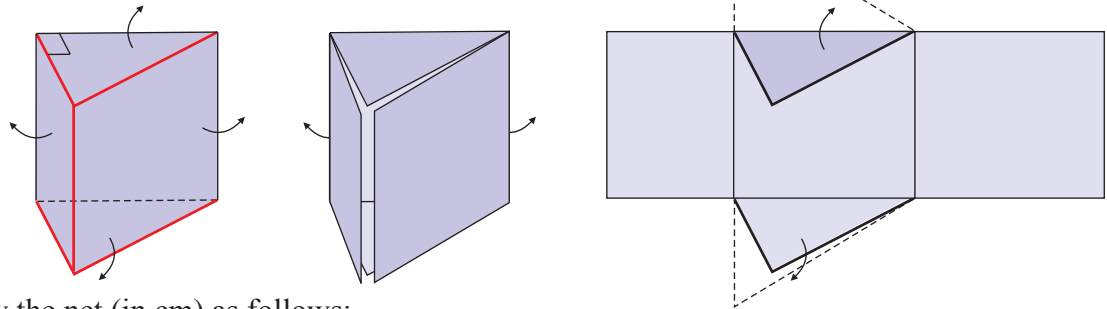
With gratitude,
Zvezdelina Stankova
Translator, Adapter, and Co-Author

10. Surface Area of a Right Prism

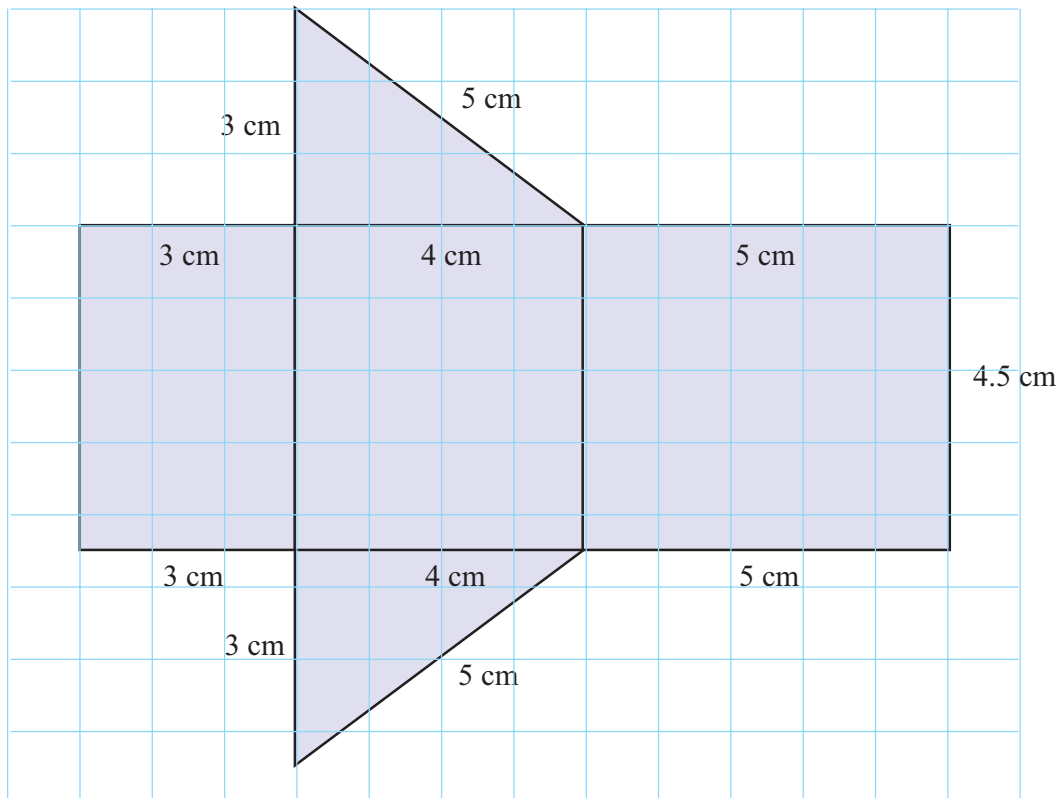
Example 1 On a square grid draw the net of a right triangular prism with height 4.5 cm whose base is a right triangle with legs 3 cm and 4 cm and hypotenuse 5 cm.

Solution:

Consider the right triangular prism drawn below. If we cut it along the colored edges and open it in the direction of the arrows to the left and to the right, and up and down, we will obtain **the net of the regular triangular prism:**



We draw the net (in cm) as follows:



D

Lateral and total surface areas of a prism

The lateral faces of a prism comprise its **lateral surface**.

The lateral faces plus the bases comprise the **(total) surface area** of the prism.

The **lateral surface area** S of the right prism is the sum of the areas of its lateral faces.

The regular prism in *Problem 1* has the lateral surface area equal to:

$$\begin{aligned} S &= 3 \cdot 4.5 + 4 \cdot 4.5 + 5 \cdot 4.5 = \\ &= \underbrace{(3 + 4 + 5)}_P \cdot \underbrace{4.5}_{l=h} = 12 \cdot 4.5 = 54, \quad S = 54 \text{ cm}^2. \end{aligned}$$

We notice that $3 + 4 + 5$ is the perimeter P of the base, and 4.5 is the lateral edge, which is equal to the height h of the regular prism. Thus, for the lateral surface area of the regular triangular prism in *Problem 1* we obtain that $S = P \cdot h$.

Reasoning the same way, we obtain in general that:

The lateral surface area of any right prism is calculated by the formula:

$$S = P \cdot h,$$

where P is the perimeter of the base, and

h is the height (= the lateral edge l) of the regular prism.

The two bases of the regular prism have equal areas B and, thus, the sum of the two areas is

$$B + B = 2 \cdot B.$$

If we add to the lateral surface area S of a right prism also the areas of the two bases, we obtain the area of the (total) surface S_1 of the regular prism, which can be found by the formula:

$$S_1 = S + 2 \cdot B.$$

Surface area of a right prism

- Formula for **the lateral surface area** $\rightarrow S = P \cdot h$
- Formula for **the total surface area** $\rightarrow S_1 = S + 2 \cdot B$

Example 2 Find the surface area of the right triangular prism in *Example 1*.

Solution:

$$S_1 = S + 2 \cdot B. \quad \text{We found that } S = 54 \text{ cm}^2.$$

The base is a right triangle with legs 4 cm and 3 cm.

$$\text{Therefore, } B = \frac{4 \cdot 3}{2} = 6 \text{ cm}^2,$$

$$S_1 = S + 2 \cdot B = 54 + 2 \cdot 6 = 66, \quad S_1 = 66 \text{ cm}^2.$$

Exercises

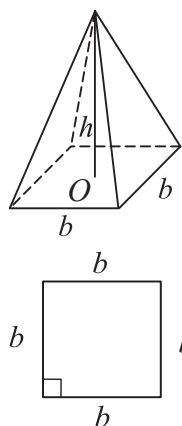
- 1 On a square grid (1 division = 1 unit) draw the net of a regular quadrilateral prism with base edge 3 units and height 4 units.
- 2 The base of a right prism is a triangle with sides $a = 5$ cm, $b = 6$ cm, and $c = 7$ cm, and the height of the prism is $h = 10$ cm. Find the lateral surface area of the prism.
- 3 The base of a right prism is a right triangle with legs $a = 6$ cm and $b = 8$ cm and hypotenuse $c = 10$ cm, and the height of the prism is 20 cm. Find the lateral and the total surface areas of the prism.
- 4 The perimeter of the base of a regular quadrilateral prism is 2 dm and its height is 80 mm. Find the lateral and the total surface areas of the prism (in cm^2).

20. Volume and Surface Area of a Regular Pyramid. Exercises

Given is a regular pyramid. In this lesson we will use the following notation:

- base edge b ,
- apothem a of the base,
- altitude h of the pyramid,
- slant height k of the pyramid,
- perimeter of the base P ,
- area of the base B ,
- lateral and total surface areas S and S_1 , and
- volume V .

Example 1 A regular quadrilateral pyramid has $S_1 = 96 \text{ cm}^2$, $h = 4 \text{ cm}$, and $P = 24 \text{ cm}$. Find S and V .



Given:
 regular pyramid
 base = square
 $P = 24 \text{ cm}$
 $h = 4 \text{ cm}$
 $S_1 = 96 \text{ cm}^2$

$S = ?$
 $V = ?$

Solution:

$$S_1 = S + B$$

$$96 = S + B \rightarrow \begin{cases} B = b^2 \rightarrow b = ? \\ P = 4 \cdot b, \\ 24 = 4 \cdot b, \\ b = 6 \text{ cm} \end{cases}$$

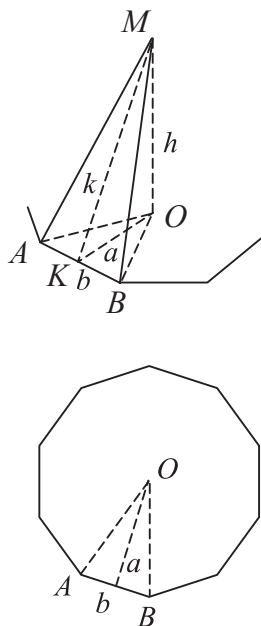
$$\begin{cases} B = 6^2 \\ B = 36 \text{ cm}^2 \end{cases}$$

$$96 = S + 36$$

$$S = 60 \text{ cm}^2$$

$$V = \frac{B \cdot h}{3} = \frac{36 \cdot 4}{3} = 48, \quad V = 48 \text{ cm}^3$$

Example 2 A regular decagonal pyramid has $S = 260 \text{ cm}^2$, $h = 6 \text{ cm}$, $P = 52 \text{ cm}$, and $a = 8 \text{ cm}$. Find S_1 , V , and k .



Given:
 regular pyramid
 base = regular
 decagon with
 $P = 52 \text{ cm}$
 $a = 8 \text{ cm}$
 $h = 6 \text{ cm}$
 $S = 260 \text{ cm}^2$

$S_1 = ?$
 $V = ?$
 $k = ?$

Solution:

$$S_1 = S + B$$

$$S_1 = 260 + B \rightarrow B = \frac{P \cdot a}{2}, \quad B = \frac{52 \cdot 8}{2}$$

$$B = 208 \text{ cm}^2$$

$$S_1 = 260 + 208 = 468$$

$$S_1 = 468 \text{ cm}^2$$

$$V = \frac{B \cdot h}{3}$$

$$V = \frac{208 \cdot 6}{3} = 208 \cdot 2 = 416$$

$$V = 416 \text{ cm}^3$$

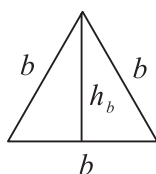
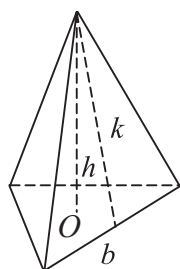
From $S = \frac{P \cdot k}{2}$ we find k .

$$260 = \frac{52 \cdot k}{2}$$

$$260 = 26 \cdot k$$

$$k = 10 \text{ cm}$$

Example 3 A regular triangular pyramid has $V = 519 \text{ cm}^3$, $h = 12 \text{ cm}$, $k = 13 \text{ cm}$, and altitude of the base $h_b = 15 \text{ cm}$. Find S and S_1 .



Given:
regular
pyramid
base = equilateral
triangle with
 $h_b = 15 \text{ cm}$
 $V = 519 \text{ cm}^3$
 $h = 12 \text{ cm}$
 $k = 13 \text{ cm}$
 $S = ?$
 $S_1 = ?$

Solution:

$$S = \frac{P \cdot k}{2}, \quad S = \frac{3 \cdot b \cdot 13}{2} \quad b = ?$$

From $V = \frac{B \cdot h}{3}$ we find B :

$$519 = \frac{B \cdot 12}{3}, \quad 519 = B \cdot 4, \quad B = 129.75 \text{ cm}^2$$

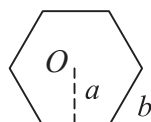
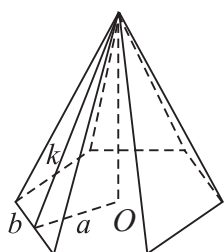
From $B = \frac{b \cdot h_b}{2}$ we find b :

$$129.75 = \frac{b \cdot 15}{2}, \quad b = 17.3 \text{ cm}$$

$$\text{Then } S = \frac{3 \cdot b \cdot 13}{2} = \frac{3 \cdot 17.3 \cdot 13}{2}, \quad S = 337.35 \text{ cm}^2.$$

$$S_1 = S + B = 337.35 + 129.75, \quad S_1 = 467.10 \text{ cm}^2$$

Example 4 A regular hexagonal pyramid has $S = 69 \text{ cm}^2$, $V = 55.2 \text{ cm}^3$, $k = 5 \text{ cm}$, and $a = 4 \text{ cm}$. Find b , h , and S_1 .



Given:
regular
pyramid
base = regular
hexagon with
 $a = 4 \text{ cm}$
 $S = 69 \text{ cm}^2$
 $V = 55.2 \text{ cm}^3$
 $k = 5 \text{ cm}$
 $b = ?$
 $h = ?$
 $S_1 = ?$

Solution:

$$b = ? \quad \text{From } S = \frac{P \cdot k}{2}, \text{ i.e., } S = \frac{6 \cdot b \cdot k}{2}, \text{ we find } b:$$

$$69 = \frac{6 \cdot b \cdot 5}{2}, \quad 69 = 15 \cdot b, \quad b = 4.6 \text{ cm}$$

$$h = ? \quad \text{From } V = \frac{B \cdot h}{3} \text{ we find } h:$$

$$55.2 = \frac{B \cdot h}{3} \rightarrow \begin{cases} B = \frac{P \cdot a}{2} = \frac{6 \cdot b \cdot 4}{2} \\ B = 12 \cdot 4.6 \\ B = 55.2 \text{ cm}^2 \end{cases}$$

$$55.2 = \frac{55.2 \cdot h}{3}, \quad h = 3 \text{ cm}$$

$$S_1 = ? \quad S_1 = S + B \quad S_1 = 69 + 55.2 = 124.2$$

$$S_1 = 124.2 \text{ cm}^2$$

Exercises

- 1 A regular quadrilateral pyramid has volume $V = 48 \text{ cm}^3$, altitude $h = 4 \text{ cm}$, and slant height $k = 5 \text{ cm}$. Find the base edge b , the lateral surface area S , and the total surface area S_1 of the pyramid.
- 2 A regular hexagonal pyramid has lateral surface area $S = 69 \text{ cm}^2$, altitude $h = 3 \text{ cm}$, slant height $k = 5 \text{ cm}$, and apothem of the base $a = 4 \text{ cm}$. Find the base edge b , the surface area S_1 , and the volume V of the pyramid.

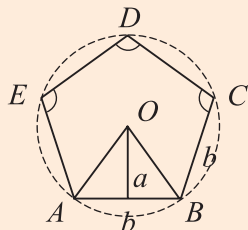
- 3 A regular pentagonal pyramid has surface area $S_1 = 522 \text{ cm}^2$, slant height $k = 10 \text{ cm}$, apothem of the base $a = 8 \text{ cm}$, and volume $V = 464 \text{ cm}^3$. Find the base edge b , the lateral surface area S , and the altitude h of the pyramid.
- 4 A regular triangular pyramid has altitude $h = 4 \text{ cm}$, volume $V = 62.4 \text{ cm}^3$, and surface area $S_1 = 124.8 \text{ cm}^2$. If the altitude of the base is $h_b = 9 \text{ cm}$, find the base edge b , the slant height k , and the lateral surface area S of the pyramid.

21.

Summary of the Topic "Polyhedral Solids"

We remember:

Regular polygon



$$AB = BC = CD = DE = EA = b$$

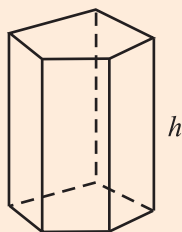
$$\angle A = \angle B = \angle C = \angle D = \angle E$$

a = the apothem

$$P = n \cdot b \quad (n \text{ is the number of sides})$$

$$B = \frac{P \cdot a}{2}$$

A right prism



Lateral faces: rectangles

Two bases: polygons

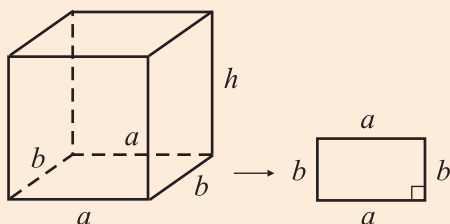
$$S = P \cdot h, \quad S_1 = S + 2 \cdot B, \quad V = B \cdot h.$$

where h is the altitude of the prism, $h = l$;
 P is the perimeter of the base;
 B is area of the base.

$$\Sigma = 2 \cdot P + n \cdot h \quad \text{is the sum of the edges of the prism.}$$

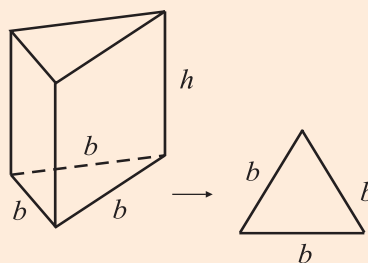
A rectangular parallelepiped

is a type of a right prism
whose bases are rectangles.

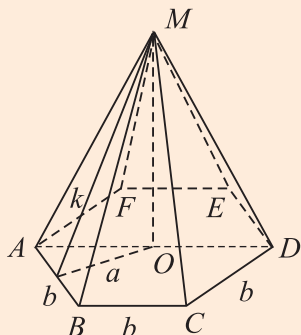


A regular prism

is a type of a prism
whose bases are regular polygons.



A regular pyramid



Lateral faces: same isosceles triangles.

The base: a regular polygon.

$$S = \frac{P \cdot k}{2}, \quad S_1 = S + B, \quad V = \frac{B \cdot h}{3}.$$

where k is slant height of the pyramid,

$$B = \frac{P \cdot a}{2}.$$

a is the apothem of the base.

$$\Sigma = P + n \cdot l$$

A geometric solid enclosed by a surface made of only polygons is called a **polyhedron**.
The polygons are called the **faces** and their sides are called the **edges** of the polyhedron.

Example 1 Given is a regular n -gon prism with base edge b , apothem of the base a , height h , perimeter of the base P , area of the base B , lateral surface area S , surface area S_1 , and volume V . Fill in the table below:

	n	b (cm)	a (cm)	P (cm)	B (cm ²)	h (cm)	S (cm ²)	S_1 (cm ²)	V (cm ³)
a)	5	11.6	8	?	?	10	?	?	?
b)	6	?	5	?	87	?	?	313.2	?
c)	12	?	8	62.4	?	?	?	?	499.2

Solution:

a)	$P = 5 \cdot b = 58$	$B = \frac{P \cdot a}{2} = 232$	$S = P \cdot h = 580$	$S_1 = S + 2 \cdot B = 1,044$	$V = B \cdot h = 2,320$
b)	$P = \frac{2 \cdot B}{a} = 34.8$	$b = P \div 6 = 5.8$	$S = S_1 - 2 \cdot B = 139.2$	$h = \frac{S}{P} = 4$	$V = B \cdot h = 384$
c)	$b = P \div 12 = 5.2$	$B = \frac{P \cdot a}{2} = 249.6$	$h = \frac{V}{B} = 2$	$S = P \cdot h = 124.8$	$S_1 = S + 2 \cdot B = 624$

Example 2 Given is a regular n -gon pyramid with base edge b and apothem of the base a . The pyramid has altitude h , slant height k , perimeter of the base P , area of the base B , lateral surface area S , surface area S_1 , and volume V . Fill in the table below:

	n	b (cm)	a (cm)	P (cm)	B (cm ²)	h (cm)	k (cm)	S (cm ²)	S_1 (cm ²)	V (cm ³)
a)	5	11.6	8	?	?	6	10	?	?	?
b)	6	?	5	?	87	12	?	?	313.2	?
c)	12	?	8	62.4	?	?	?	312	?	499.2

Solution:

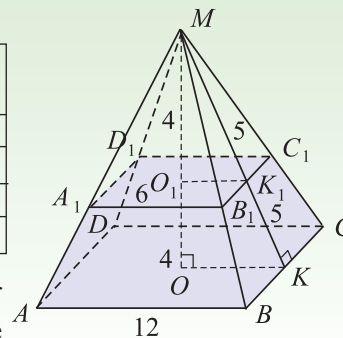
a)	$P = 5 \cdot b = 58$	$B = \frac{P \cdot a}{2} = 232$	$S = \frac{P \cdot k}{2} = 290$	$S_1 = S + B = 522$	$V = \frac{B \cdot h}{3} = 464$
b)	$P = \frac{2 \cdot B}{a} = 34.8$	$b = P \div 6 = 5.8$	$S = S_1 - B = 226.2$	$k = \frac{2 \cdot S}{P} = 13$	$V = \frac{B \cdot h}{3} = 348$
c)	$b = P \div 12 = 5.2$	$B = \frac{P \cdot a}{2} = 249.6$	$h = \frac{3 \cdot V}{B} = 6$	$k = \frac{2 \cdot S}{P} = 10$	$S_1 = S + B = 561.6$

Exercises

1 Fill the table below for a *regular hexagonal prism* with base edge b , apothem of the base a , perimeter of the base P , and area of the base B . The altitude of the prism is h , the lateral surface S , the surface area S_1 , the volume V , and the sum of all edges Σ .

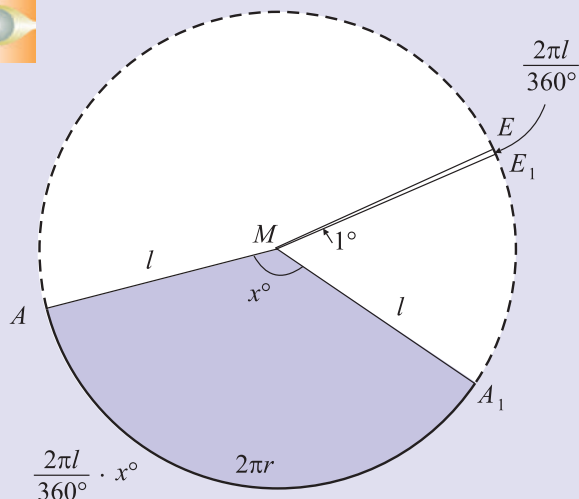
	b (cm)	a (cm)	P (cm)	B (cm ²)	h (cm)	S (cm ²)	S_1 (cm ²)	V (cm ³)	Σ (cm)
a)	4.6	4	?	?	10	?	?	?	?
b)	?	?	21	31.5	?	126	?	?	?
c)	?	?	34.8	87	?	?	348	?	?
d)	?	3.5	?	?	10	?	?	420	?

2 From a regular quadrilateral pyramid we have cut out a smaller regular quadrilateral pyramid, as shown to the right. Using the measures in the drawing (in cm), find the volume and the surface area of the resulting solid.



27. Surface Area of a Cone

Example 1 Draw the net of a cone of radius $r = 2$ cm and generatrix $l = 5$ cm.



We complete the conical surface (sector AMA_1) to a whole disk, as shown in the drawing. This disk has circumference $2\pi \cdot l$.

We know that the angle at center M of the disk is 360° .

To every circular sector with angle 1° corresponds an arc $\widehat{EE_1}$ with length $\frac{2 \cdot \pi \cdot l}{360}$; i.e., $\widehat{EE_1} = \frac{2 \cdot \pi \cdot l}{360}$.

To every circular sector with angle x° corresponds an arc $\widehat{AA_1}$ with length $\frac{2 \cdot \pi \cdot l}{360} \cdot x$: $\widehat{AA_1} = \frac{2 \cdot \pi \cdot l}{360} \cdot x$.

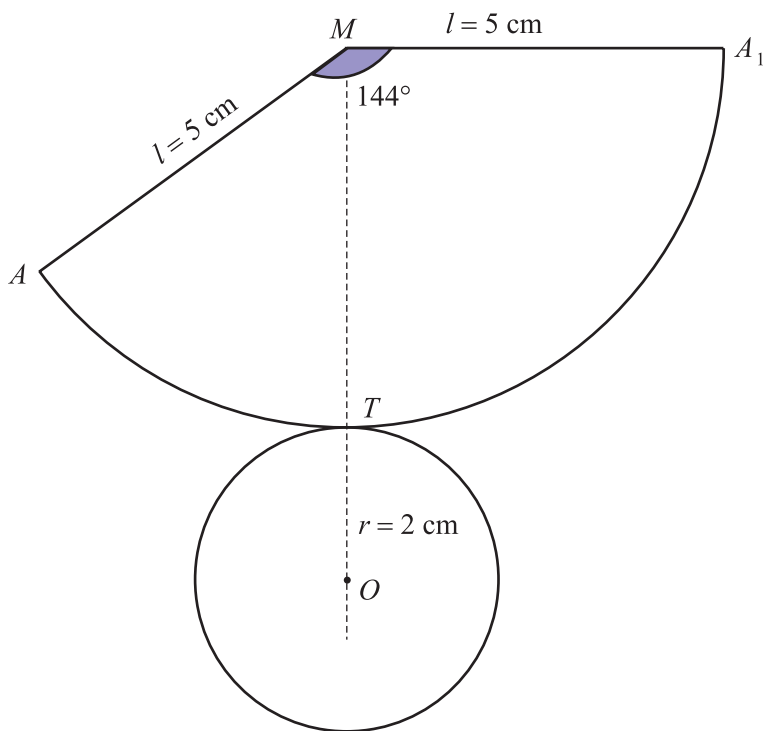
On the other hand, $\widehat{AA_1} = 2 \cdot \pi \cdot r$ is the circumference of the disk that is the base of the cone. Hence,

$$\frac{2 \cdot \pi \cdot l}{360} \cdot x = 2 \cdot \pi \cdot r \text{ and } x = \frac{r}{l} \cdot 360.$$

Thus, $x^\circ = \frac{r}{l} \cdot 360^\circ$ is the central angle that corresponds to arc $\widehat{AA_1}$. We obtain that the central angle of the sector representing the lateral surface of a cone is found by the formula

(1)

$$x^\circ = \frac{r}{l} \cdot 360^\circ.$$



Solution:

With formula (1) above we can calculate the angle made by the two generatrices, and then we can draw the conical surface of a cone given r and l .

Thus, we calculate $\angle AMA_1 = x^\circ$:

$$x^\circ = \frac{r}{l} \cdot 360^\circ = \frac{2}{5} \cdot 360^\circ = 144^\circ.$$

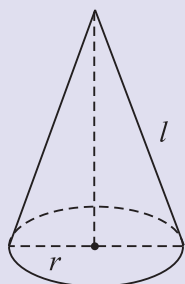
We draw:

1. Circular sector AMA_1 with $l = 5$ cm and $\angle AMA_1 = 144^\circ$ (in addition to a ruler, we use a compass and a protractor);
2. Arbitrary line through M that intersects $\widehat{AA_1}$ in point T ;
3. Point O on line MT so that $OT = 2$ cm;
4. Circle with center O and $r = 2$ cm.

The circular sector AA_1M is the conical surface of a cone of radius r and generatrix l .
 The area of the circular sector AA_1M coincides with
the lateral surface area S of the given cone.



How to find the lateral surface area S of a cone with given r and l ?



$$S_{\text{cone}} = S_{\text{circular sector}}$$

Every circular sector with angle 1° has area $\frac{\pi \cdot l^2}{360}$.

Therefore, every circular sector with angle x° has area $\frac{\pi \cdot l^2}{360} \cdot x$.

To find the area of the circular sector that is the lateral surface of a cone with generatrix l and radius r , we substitute x with the expression $x = \frac{r}{l} \cdot 360$:

$$\frac{\pi \cdot l^2}{360} \cdot x = \frac{\pi \cdot l \cdot l}{360} \cdot \frac{r}{l} \cdot 360 = \pi \cdot r \cdot l.$$

Thus, for the lateral surface area S of a right circular cone we obtain the formula **$S = \pi \cdot r \cdot l$** .

The lateral surface area of a right circular cone is found by the formula

$$S = \pi \cdot r \cdot l,$$

where r is the radius of the cone and l is the generatrix (or the slant height) of the cone.

If we add the area of the base B to the lateral surface area S , we obtain the total surface area of the cone $S_1 = S + B$.

The surface area of a right circular cone is found by the formula

$$S_1 = S + B,$$

where $S = \pi \cdot r \cdot l$ and $B = \pi \cdot r^2$ is the area of the base.

The formula $S = \pi \cdot r \cdot l$ can be written as $S = \frac{2\pi \cdot r \cdot l}{2}$ or $S = \frac{P \cdot l}{2}$, where the perimeter P of the base is the circumference $2\pi \cdot r$ of the circle.

We see that both the formulas

$$S = \frac{P \cdot l}{2} \text{ and } S_1 = S + B$$

calculate the lateral and the full surface areas of both a right circular cone and of a regular pyramid, provided $l = k$; i.e., the generatrix of the cone is equal to the slant height of the pyramid.

Exercises

1 Find in degrees the angle x° between the two generatrices of the conical surface if:

- a) $r = 3$ cm and $l = 10$ cm;
- b) $r = 4$ cm and $l = 9$ cm;
- c) $r = 1$ cm and $l = 6$ cm.

2 Draw the net of a right circular cone if:

- a) $r = 3$ cm and $l = 6$ cm;
- b) $r = 2$ cm and $l = 6$ cm;
- c) $r = 1$ cm and $l = 6$ cm;
- d) $r = 3$ cm and $l = 10$ cm.