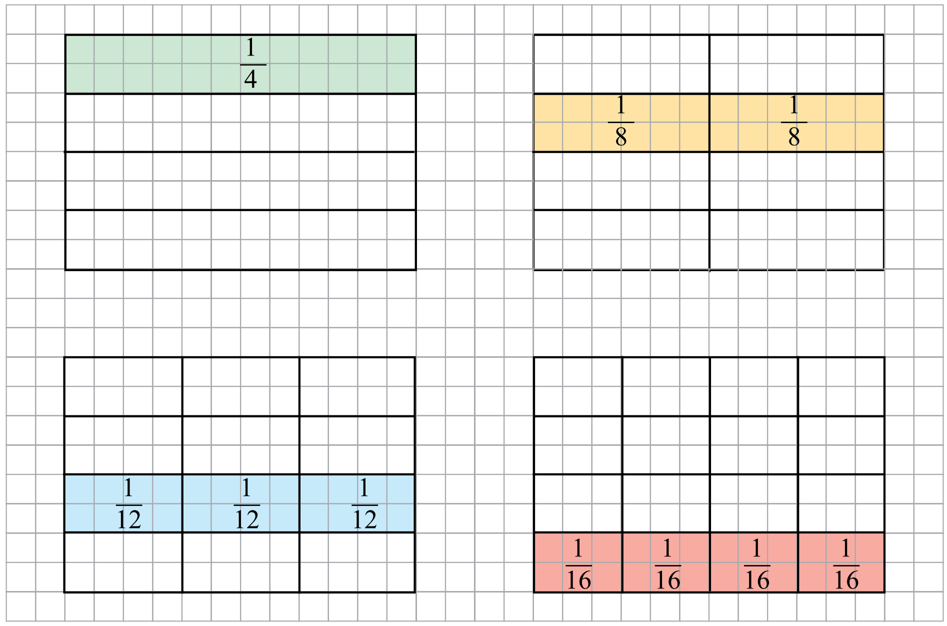
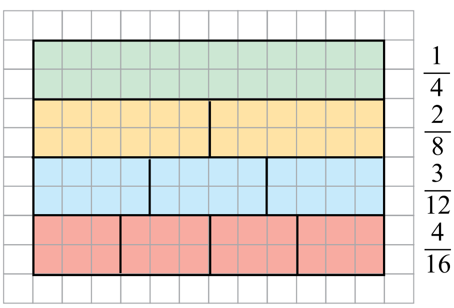
CONTENTS

93. Basic Property of Fractions. Expanding Fractions

The four rectangles in the drawing on the right have dimensions 12 and 8 (units) and are divided into 4, 8, 12, and 16 identical rectangles. Compare the areas of the colored stripes.

The areas of the colored stripes are equal to $\frac{1}{4}$ of the area of the rectangle and they make (larger) rectangles with same areas:





$$\rightarrow \quad \frac{1}{4}=\frac{2}{8}=\frac{3}{12}=\frac{4}{16}.$$

$$\frac{1}{4}=\frac{1\cdot 2}{4\cdot 2}=\frac{2}{8} \quad \rightarrow \quad \frac{2}{8}=\frac{2\div 2}{8\div 2}=\frac{1}{4},$$

$$\frac{1}{4}=\frac{1\cdot 3}{4\cdot 3}=\frac{3}{12} \quad \rightarrow \quad \frac{3}{12}=\frac{3\div 3}{12\div 3}=\frac{1}{4},$$

$$\frac{1}{4}=\frac{1\cdot 4}{4\cdot 4}=\frac{4}{16} \quad \rightarrow \quad \frac{4}{16}=\frac{4\div 4}{16\div 4}=\frac{1}{4}.$$

The equality $\frac{1}{4}=\frac{2}{8}$ $\left(\frac{2}{8}=\frac{1}{4}\right)$ shows that:

- when multiplying the numerator and the denominator of $\frac{1}{4}$ by 2, or
- when dividing the numerator and the denominator of $\frac{2}{8}$ by 2,

we obtain a fraction equal to the original.

Basic property of fractions

If the numerator and the denominator of a fraction are multiplied (divided) by a number different from 0, we obtain a fraction equal to the original:

$$\frac{a}{b}=\frac{a\cdot n}{b\cdot n}, \quad n\neq 0; \qquad \frac{a}{b}=\frac{a\div m}{b\div m}, \quad m\neq 0.$$

Examples: $\frac{2}{5}=\frac{2\cdot 3}{5\cdot 3}=\frac{6}{15}$; We say that we have **expanded** the fraction $\frac{2}{5}$ by 3.

$\frac{6}{15}=\frac{6\div 3}{15\div 3}=\frac{2}{5}$. We say that we have **reduced** the fraction $\frac{6}{15}$ by 3.

If the number $n = 1$ (or $m = 1$), then the fraction does *not* change.

Example 1

Expand the fractions by the number 2: a) $\frac{5}{6}$; b) $\frac{11}{15}$; c) $\frac{20}{17}$.

Solution:

a) $\frac{5}{6} = \frac{5 \cdot 2}{6 \cdot 2} = \frac{10}{12}$; b) $\frac{11}{15} = \frac{11 \cdot 2}{15 \cdot 2} = \frac{22}{30}$; c) $\frac{20}{17} = \frac{20 \cdot 2}{17 \cdot 2} = \frac{40}{34}$.

The number by which we expand the fraction is called **an extra factor**, which we usually write above the fraction.

$$\frac{5}{6} = \frac{5 \cdot 2}{6 \cdot 2} = \frac{10}{12} \longrightarrow \boxed{\overset{2}{\frac{5}{6}} = \frac{10}{12}}$$

Example 2

Expand the fractions: a) $\frac{7}{12}$ by 2; b) $\frac{3}{2}$ by 12; c) $\frac{1}{4}$ by 6.

Solution:

a) $\overset{2}{\frac{7}{12}} = \frac{14}{24}$; b) $\overset{12}{\frac{3}{2}} = \frac{36}{24}$; c) $\overset{6}{\frac{1}{4}} = \frac{6}{24}$.

When expanding the fractions in *Example 2* we obtained fractions with equal denominators.

Example 3

Find the extra factors of the fraction $\frac{7}{8}$ if:

a) $\frac{7}{8} = \frac{14}{16}$; b) $\frac{7}{8} = \frac{35}{40}$; c) $\frac{7}{8} = \frac{84}{96}$.

Solution:

a) $\overset{2}{\frac{7}{8}} = \frac{14}{16}$; b) $\overset{5}{\frac{7}{8}} = \frac{35}{40}$; c) $\overset{12}{\frac{7}{8}} = \frac{84}{96}$.

The extra factor in *Example 3* is the number by which we reduce:

$$\frac{14}{16} = \frac{14 \div 2}{16 \div 2} = \frac{7}{8}.$$

Example 4

Expand the fractional part of the mixed number $13\frac{2}{3}$ by:

a) 2; b) 3; c) 4; d) 5.

Solution:

a) $13\frac{2}{3} = 13\frac{2 \cdot 2}{3 \cdot 2} = 13\frac{4}{6}$; b) $13\frac{6}{9}$; c) $13\frac{8}{12}$; d) $13\frac{10}{15}$.

Exercises

1 Expand the fractions $\frac{3}{5}$, $\frac{4}{7}$, $\frac{5}{8}$ and $\frac{11}{13}$:

a) by 2; b) by 3; c) by 5; d) by 7.

2 Expand the fraction:

a) $\frac{1}{3}$ by 10; b) $\frac{2}{5}$ by 19; c) $\frac{6}{7}$ by 22.

3 Find the missing numbers:

a) $\frac{3}{4} = \frac{?}{12}$; c) $\frac{3}{4} = \frac{?}{20}$;

b) $\frac{3}{4} = \frac{15}{?}$; d) $\frac{3}{4} = \frac{27}{?}$.

4 Expand the fractional part of the mixed numbers by 2:

a) $3\frac{1}{3}$, $5\frac{2}{7}$; c) $15\frac{11}{17}$, $27\frac{13}{19}$;

b) $3\frac{5}{9}$, $7\frac{11}{13}$; d) $8\frac{5}{12}$, $16\frac{3}{11}$.

5 Expand the fractions to get equal denominators:

a) $\frac{5}{9}$ and $\frac{2}{3}$; c) $\frac{5}{12}$ and $\frac{2}{3}$;

b) $\frac{3}{10}$ and $\frac{2}{5}$; d) $\frac{7}{30}$ and $\frac{4}{15}$.

118. Part of a Number. Basic Exercises

Example 1 In March of 2006 Mr. Nikolov:

	Solution:
a • had a monthly income of \$6,000; • spent $\frac{3}{5}$ of it for food. **How much money did he spend on food?**	**a** \$$x$ were spent on food. $\frac{3}{5}$ of $6{,}000 = x$ $\frac{3}{5} \cdot 6{,}000 = x$ $x = 3{,}600$ He spent \$3,600 on food.
b • spent \$3,600 on food; • spent $\frac{3}{5}$ of his monthly income on food. **How much is his monthly income?**	**b** \$$x$ is his monthly income. $\frac{3}{5}$ of $x = 3{,}600$ $\frac{3}{5} \cdot x = 3{,}600$ $x = 3{,}600 \div \frac{3}{5}$ $x = 6{,}000$ His monthly income is \$6,000.
c • had a monthly income of \$6,000; • spent \$3,600 on food. **What part of his monthly income did he spent on food?**	**c** x parts x of $6{,}000 = 3{,}600$ $x = 3{,}600 \div 6{,}000$ $x = \frac{3{,}600}{6{,}000}$ $x = \frac{3}{5}$ He spent $\frac{3}{5}$ of his income on food.

We notice that the following quantities play major roles in *Example 1*:

- the monthly income (\$6,000);
- the amount for food (\$3,600);
- part of the monthly income spent on food $\left(\frac{3}{5}\right)$.

These quantities satisfy the following relationship:

$$\frac{3}{5} \text{ of } 6{,}000 = 3{,}600, \quad \text{i.e.,} \quad \frac{3}{5} \cdot 6{,}000 = 3{,}600.$$

Remember that p of $a = b$ is the same as $p \cdot a = b$

We solved three types of problems:

Problems	Given	We are looking for:
Type 1 (as in part a)	$\frac{3}{5}$ and 6,000	3,600 → part of the given number
Type 2 (as in part b)	$\frac{3}{5}$ and 3,600	6,000 → a number given a part of it
Type 3 (as in part c)	3,600 and 6,000	$\frac{3}{5}$ → what part of 6,000 is 3,600

Problems of Type I: Find x if $\boxed{\frac{1}{3} \text{ of } 18 = x}$. We solve: $\frac{1}{3} \cdot 18 = x$, $x = 6$.

Example 2 The elementary school students in one school are 720. $\frac{1}{4}$ of them are in 5th grade. How many fifth graders are there in this school?

Solution: There are x students in 5th grade in this school.

$$x = \frac{1}{4} \text{ of } 720, \qquad x = \frac{1}{4} \cdot 720, \qquad x = 180.$$

Problems of Type II: Find x if $\boxed{\frac{1}{3} \text{ of } x = 6}$. We solve: $\frac{1}{3} \cdot x = 6$, $x = 18$.

Example 3 A student read 56 pages in one day, which is $\frac{2}{5}$ of a book. How many pages does the book have?

Solution: The book has x pages.

$$\frac{2}{5} \text{ of } x = 56, \qquad \frac{2}{5} \cdot x = 56, \qquad x = 56 \div \frac{2}{5}, \qquad x = 140.$$

The book has 140 pages.

Problems of Type III: Find x if $\boxed{x \text{ of } 18 = 6}$. We solve: $x \cdot 18 = 6$, $x = \frac{1}{3}$.

Example 4 As a preparation for the second round of a math olympiad, a teacher assigned 120 problems. Angel managed to solve 80 of them by himself . What part of all of the problems did Angel solve?

Solution: x part of all problems were solved by Angel.

$$x \text{ of } 120 = 80, \qquad x \cdot 120 = 80, \qquad x = 80 \div 120, \qquad x = \frac{80}{120}, \qquad x = \frac{2}{3}.$$

Angel solved $\frac{2}{3}$ of the problems by himself .

Exercises

1 Find x if:

a) $\frac{2}{5}$ of $x = 30$; b) $\frac{3}{8}$ of $x = 6.3$.

2 The tractor operators in a cooperative plowed 800 dec in one day, which is $\frac{2}{7}$ of the total arable area. How many decares of arable area does this cooperative own?

3 A trader re-sold 105 crates of grapes, which is $\frac{3}{4}$ of the total crates he had originally bought at a market. How many crates of grapes had the trader bought?

4 Find x if:

a) x of $45 = 36$; c) x of $12.6 = 8.4$;

b) x of $72 = 45$; d) x of $2\frac{1}{3} = \frac{2}{3}$.

5 Ivan saved \$370 for a bike. The price of the bike is \$444.

a) What part of the price did Ivan save?

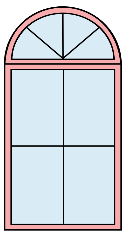
b) What part of the price does Ivan still have to save?

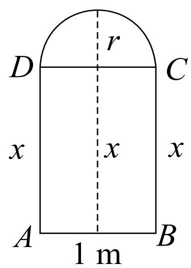
6 Out of assigned 85 problems Peter managed to solve 68 problems by the deadline. What part of the problems did Peter solve?

7 Dmitri took a test in his specialty. From the given 90 problems he solved 75. To pass the test he needed to solve $\frac{5}{6}$ of all problems. Did Dmitri pass the test?

174. Length of a Circle. Practical Exercises

Example 1 The frame of a window is in the shape of a rectangle and a semi-circle glued together as shown and has a perimeter of 5.37 m. Find the height of window, if its width is 1 m.





Solution:

We denote the rectangular part of the window by $ABCD$.

Thus, $AB = CD = 1$ m.

$AD = BC = x$.

CD is the diameter of the semi-circle, and

$$r = \frac{1}{2} \cdot CD = \frac{1}{2} \cdot 1 = 0.5 \text{ m.}$$

The length of the semi-circle is

$$\frac{1}{2} \text{ of } c = \frac{1}{2} \cdot 2 \cdot \pi \cdot r = \pi \cdot r = 3.14 \cdot 0.5 = 1.57 \text{ m.}$$

From the equality $\frac{1}{2} \cdot c + AB + 2 \cdot x = 5.37$ we find the unknown x:

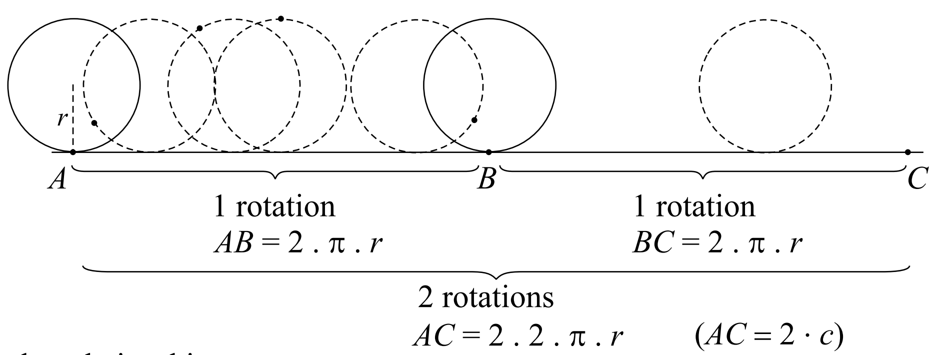
$$\begin{aligned}
1.57 + 1 + 2 \cdot x &= 5.37 \\
2.57 + 2 \cdot x &= 5.37 \\
2 \cdot x &= 5.37 - 2.57 \\
2 \cdot x &= 2.80 \\
x &= 2.80 \div 2 \\
x &= 1.40.
\end{aligned}$$

The height of the window is obtained from $x + r = 1.40 + 0.50 = 1.90$.

The height of the window is 1.90 m.

Example 2 A gymnast works with a hoop that has a diameter of 80 cm. How many meters will the hoop travel if it makes 2 full rotations? How about 10 full rotations?

Solution:



We obtain the relationship:

the distance s = number of rotations · the length of the circle.

The length of the circle is thus calculated by:

$$c = 3.14 \cdot 0.8 = 2.512 \text{ m} \quad (80 \text{ cm} = 0.8 \text{ m}).$$

With 2 full rotations $\rightarrow AC = 2 \cdot c = 2 \cdot 2.512 = 5.024$ m.

With 10 full rotations $\rightarrow 10 \cdot c = 10 \cdot 2.512 = 25.12$ m.

Example 3 A mother is pushing a stroller with her small child. If the front wheel has radius 12 cm, can you find how many times it will have turned after the mother walks 200 meters?

Solution:

The wheel has made x full turns. The distance traveled by it is 200 m. From the relationship

number of turns $\cdot\, c = s$ (the distance traveled)

we find that

$$x \cdot 2 \cdot \pi \cdot r = 20{,}000 \text{ (in cm)}$$
$$x \cdot 6.28 \cdot 12 = 20{,}000$$
$$x = 20{,}000 \div 75.36$$
$$x = 265.39.$$

The number of full turns is 265.

Example 4 The tire of a car has radius 40 cm. What is the speed of the car if the tire makes 600 revolutions in one minute?

Solution:

The speed of the car is the number of kilometers which it covers in 1 hour (= 60 minutes).

1 revolution = 1 full turn of the wheel; i.e.,

1 revolution → $c = 2 \cdot \pi \cdot 40 = 251.2$ cm.

In 1 minute it makes 600 revolutions → $600 \cdot 251.2 = 150{,}720$ cm.

In 1 hour it makes $60 \cdot 600$ revolutions → $60 \cdot 150\,720 = 9{,}043{,}200$ cm.

9,043,200 cm = 90,432 m = 90.432 km ≈ 90.43 km.

In 1 hour the tire covers a distance of 90.43 km; i.e., the speed of the car is $v = 90.43$ km/h.

The ancient Babylonians believed that the ratio of the length of every circle and the length of its diameter is the number 3. The Egyptians used the number 3.16.

Archimedes (287-212 BCE) determined the number π with accuracy to 2 decimal digits.

Ludolph Van Ceulen (1540-1610) introduced the notation π (pi) for this number.

Nowadays, with the help of computers, π can be determined up to trillions of decimal digits:

$$\pi = 3.14159265358979323846\ldots$$

Exercises

1. A round mirror has a diameter of 80 cm. Find out how long its frame is in meters.
2. You have a large piece of (flat) glass and you want to cover with it a round table with diameter 1 m. To cut the glass, the handyman asks for $12, and to smooth out the edge he charges $1.20 per 10 cm. How many dollars do you have to pay?
3. Ani rolls a hoop with a diameter of 80 cm. If the hoop makes 5 full turns, how many meters does it travel?
4. Milan biked 150 m. How many times did the wheels of the bike turn if both have a diameter of 56 cm?
5. The tire of a car has a radius of 40 cm. With what speed does the car travel if the tire makes 550 revolutions in a minute?
6. The wheel of a roadtanker has a diameter of 180 cm. In 5 minutes it makes 1,000 revolutions. What is the speed of the roadtanker?