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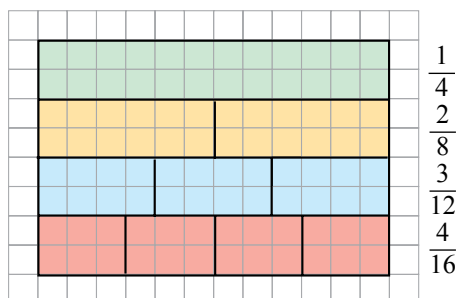
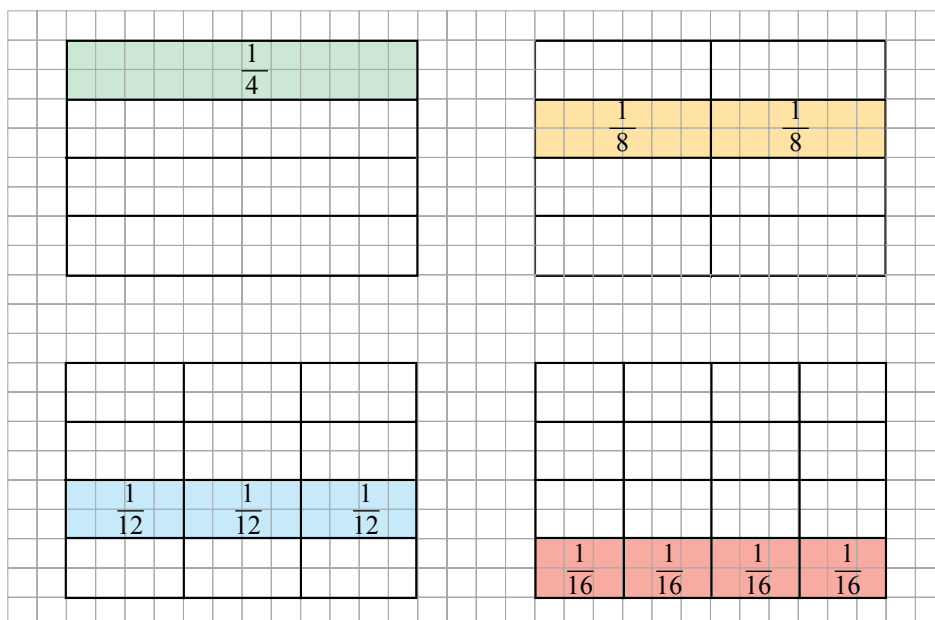
Credits ..... XLI

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# 93. Basic Property of Fractions. Expanding Fractions

The four rectangles in the drawing on the right have dimensions 12 and 8 (units) and are divided into 4, 8, 12, and 16 identical rectangles. Compare the areas of the colored stripes.

The areas of the colored stripes are equal to  $\frac{1}{4}$  of the area of the rectangle and they make (larger) rectangles with same areas:



$$\begin{aligned} \frac{1}{4} &= \frac{2}{8} = \frac{3}{12} = \frac{4}{16} \cdot \\ \frac{1}{4} &= \frac{1 \cdot 2}{4 \cdot 2} = \frac{2}{8} \rightarrow \frac{2}{8} = \frac{2 \div 2}{8 \div 2} = \frac{1}{4}, \\ \frac{1}{4} &= \frac{1 \cdot 3}{4 \cdot 3} = \frac{3}{12} \rightarrow \frac{3}{12} = \frac{3 \div 3}{12 \div 3} = \frac{1}{4}, \\ \frac{1}{4} &= \frac{1 \cdot 4}{4 \cdot 4} = \frac{4}{16} \rightarrow \frac{4}{16} = \frac{4 \div 4}{16 \div 4} = \frac{1}{4}. \end{aligned}$$

The equality  $\frac{1}{4} = \frac{2}{8} = \left(\frac{2}{8} = \frac{1}{4}\right)$  shows that:

- when multiplying the numerator and the denominator of  $\frac{1}{4}$  by 2, or
- when dividing the numerator and the denominator of  $\frac{2}{8}$  by 2, we obtain a fraction equal to the original.

## Basic property of fractions

If the numerator and the denominator of a fraction are multiplied (divided) by a number different from 0, we obtain a fraction equal to the original:

$$\frac{a}{b} = \frac{a \cdot n}{b \cdot n}, \quad n \neq 0; \quad \frac{a}{b} = \frac{a \div m}{b \div m}, \quad m \neq 0.$$

Examples:  $\frac{2}{5} = \frac{2 \cdot 3}{5 \cdot 3} = \frac{6}{15};$

We say that we have **expanded** the fraction  $\frac{2}{5}$  by 3.

$$\frac{6}{15} = \frac{6 \div 3}{15 \div 3} = \frac{2}{5}.$$

We say that we have **reduced** the fraction  $\frac{6}{15}$  by 3.

If the number  $n = 1$  (or  $m = 1$ ), then the fraction does *not* change.

**Example 1** Expand the fractions by the number 2: a)  $\frac{5}{6}$ ; b)  $\frac{11}{15}$ ; c)  $\frac{20}{17}$ .

Solution:

$$\text{a) } \frac{5}{6} = \frac{5 \cdot 2}{6 \cdot 2} = \frac{10}{12}; \quad \text{b) } \frac{11}{15} = \frac{11 \cdot 2}{15 \cdot 2} = \frac{22}{30}; \quad \text{c) } \frac{20}{17} = \frac{20 \cdot 2}{17 \cdot 2} = \frac{40}{34}$$

The number by which we expand the fraction is called an **extra factor**, which we usually write above the fraction.

$$\frac{5}{6} = \frac{5 \cdot 2}{6 \cdot 2} = \frac{10}{12} \rightarrow \boxed{\frac{2}{6} = \frac{10}{12}}$$

**Example 2** Expand the fractions: a)  $\frac{7}{12}$  by 2; b)  $\frac{3}{2}$  by 12; c)  $\frac{1}{4}$  by 6.

Solution:

$$\text{a) } \frac{7}{12} = \frac{14}{24}; \quad \text{b) } \frac{3}{2} = \frac{36}{24}; \quad \text{c) } \frac{1}{4} = \frac{6}{24}$$



When expanding the fractions in *Example 2* we obtained fractions with equal denominators.

**Example 3** Find the extra factors of the fraction  $\frac{7}{8}$  if:

$$\text{a) } \frac{7}{8} = \frac{14}{16}; \quad \text{b) } \frac{7}{8} = \frac{35}{40}; \quad \text{c) } \frac{7}{8} = \frac{84}{96}$$

Solution:

$$\text{a) } \frac{7}{8} = \frac{14}{16}; \quad \text{b) } \frac{7}{8} = \frac{35}{40}; \quad \text{c) } \frac{7}{8} = \frac{84}{96}$$



The extra factor in *Example 3* is the number by which we reduce:

$$\frac{14}{16} = \frac{14 \div 2}{16 \div 2} = \frac{7}{8}$$

**Example 4** Expand the fractional part of the mixed number  $13\frac{2}{3}$  by:

a) 2; b) 3; c) 4; d) 5.

Solution:

$$\text{a) } 13\frac{2}{3} = 13\frac{2 \cdot 2}{3 \cdot 2} = 13\frac{4}{6}; \quad \text{b) } 13\frac{6}{9}; \quad \text{c) } 13\frac{8}{12}; \quad \text{d) } 13\frac{10}{15}$$

## Exercises

**1** Expand the fractions  $\frac{3}{5}$ ,  $\frac{4}{7}$ ,  $\frac{5}{8}$  and  $\frac{11}{13}$ :

a) by 2; b) by 3; c) by 5; d) by 7.

**2** Expand the fraction:

a)  $\frac{1}{3}$  by 10; b)  $\frac{2}{5}$  by 19; c)  $\frac{6}{7}$  by 22.

**3** Find the missing numbers:

$$\text{a) } \frac{3}{4} = \frac{?}{12}; \quad \text{c) } \frac{3}{4} = \frac{?}{20};$$

$$\text{b) } \frac{3}{4} = \frac{15}{?}; \quad \text{d) } \frac{3}{4} = \frac{27}{?}$$

**4** Expand the fractional part of the mixed numbers by 2:

$$\text{a) } 3\frac{1}{3}, 5\frac{2}{7}; \quad \text{c) } 15\frac{11}{17}, 27\frac{13}{19};$$

$$\text{b) } 3\frac{5}{9}, 7\frac{11}{13}; \quad \text{d) } 8\frac{5}{12}, 16\frac{3}{11}$$

**5** Expand the fractions to get equal denominators:

$$\text{a) } \frac{5}{9} \text{ and } \frac{2}{3}; \quad \text{c) } \frac{5}{12} \text{ and } \frac{2}{3};$$

$$\text{b) } \frac{3}{10} \text{ and } \frac{2}{5}; \quad \text{d) } \frac{7}{30} \text{ and } \frac{4}{15}$$

# 118. Part of a Number. Basic Exercises

**Example 1** In March of 2006 Mr. Nikolov:

- a**
- had a monthly income of \$6,000;
  - spent  $\frac{3}{5}$  of it for food.

**How much money did he spend on food?**

**Solution:**

- a** \$x were spent on food.

$$\frac{3}{5} \text{ of } 6,000 = x$$

$$\frac{3}{5} \cdot 6,000 = x$$

$$x = 3,600$$

He spent \$3,600 on food.

- b**
- spent \$3,600 on food;
  - spent  $\frac{3}{5}$  of his monthly income on food.

**How much is his monthly income?**

- b** \$x is his monthly income.

$$\frac{3}{5} \text{ of } x = 3,600$$

$$\frac{3}{5} \cdot x = 3,600$$

$$x = 3,600 \div \frac{3}{5}$$

$$x = 6,000$$

His monthly income is \$6,000.

- c**
- had a monthly income of \$6,000;
  - spent \$3,600 on food.

**What part of his monthly income did he spend on food?**

- c** x parts

$$x \text{ of } 6,000 = 3,600$$

$$x = 3,600 \div 6,000$$

$$x = \frac{3,600}{6,000}$$

$$x = \frac{3}{5}$$

He spent  $\frac{3}{5}$  of his income on food.

We notice that the following quantities play major roles in *Example 1*:

- the monthly income (\$6,000);
- the amount for food (\$3,600);
- part of the monthly income spent on food  $\left(\frac{3}{5}\right)$ .

These quantities satisfy the following relationship:

$$\frac{3}{5} \text{ of } 6,000 = 3,600, \text{ i.e., } \frac{3}{5} \cdot 6,000 = 3,600.$$

**Remember that  $p$  of  $a = b$  is the same as  $p \cdot a = b$**

We solved three types of problems:

Problems	Given	We are looking for:
Type 1 (as in part <b>a</b> )	$\frac{3}{5}$ and 6,000	3,600 → part of the given number
Type 2 (as in part <b>b</b> )	$\frac{3}{5}$ and 3,600	6,000 → a number given a part of it
Type 3 (as in part <b>c</b> )	3,600 and 6,000	$\frac{3}{5}$ → what part of 6,000 is 3,600

**Problems of Type I:** Find  $x$  if  $\frac{1}{3}$  of 18 =  $x$ . We solve:  $\frac{1}{3} \cdot 18 = x$ ,  $x = 6$ .

**Example 2** The elementary school students in one school are 720.  $\frac{1}{4}$  of them are in 5<sup>th</sup> grade. How many fifth graders are there in this school?

**Solution:** There are  $x$  students in 5<sup>th</sup> grade in this school.

$$x = \frac{1}{4} \text{ of } 720, \quad x = \frac{1}{4} \cdot 720, \quad x = 180.$$

**Problems of Type II:** Find  $x$  if  $\frac{1}{3}$  of  $x = 6$ . We solve:  $\frac{1}{3} \cdot x = 6$ ,  $x = 18$ .

**Example 3** A student read 56 pages in one day, which is  $\frac{2}{5}$  of a book. How many pages does the book have?

**Solution:** The book has  $x$  pages.

$$\frac{2}{5} \text{ of } x = 56, \quad \frac{2}{5} \cdot x = 56, \quad x = 56 \div \frac{2}{5}, \quad x = 140.$$

The book has 140 pages.

**Problems of Type III:** Find  $x$  if  $x$  of 18 = 6. We solve:  $x \cdot 18 = 6$ ,  $x = \frac{1}{3}$ .

**Example 4** As a preparation for the second round of a math olympiad, a teacher assigned 120 problems. Angel managed to solve 80 of them by himself. What part of all of the problems did Angel solve?

**Solution:**  $x$  part of all problems were solved by Angel.

$$x \text{ of } 120 = 80, \quad x \cdot 120 = 80, \quad x = 80 \div 120, \quad x = \frac{80}{120}, \quad x = \frac{2}{3}.$$

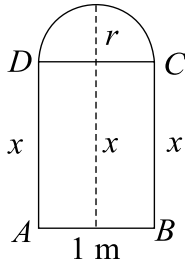
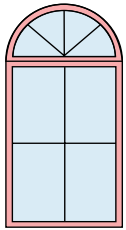
Angel solved  $\frac{2}{3}$  of the problems by himself.

## Exercises

- 1** Find  $x$  if:  
a)  $\frac{2}{5}$  of  $x = 30$ ;      b)  $\frac{3}{8}$  of  $x = 6.3$ .
- 2** The tractor operators in a cooperative plowed 800 dec in one day, which is  $\frac{2}{7}$  of the total arable area. How many decares of arable area does this cooperative own?
- 3** A trader re-sold 105 crates of grapes, which is  $\frac{3}{4}$  of the total crates he had originally bought at a market. How many crates of grapes had the trader bought?
- 4** Find  $x$  if:  
a)  $x$  of 45 = 36;      c)  $x$  of 12.6 = 8.4;
- b)  $x$  of 72 = 45;      d)  $x$  of  $2\frac{1}{3} = \frac{2}{3}$ .
- 5** Ivan saved \$370 for a bike. The price of the bike is \$444.  
a) What part of the price did Ivan save?  
b) What part of the price does Ivan still have to save?
- 6** Out of assigned 85 problems Peter managed to solve 68 problems by the deadline. What part of the problems did Peter solve?
- 7** Dmitri took a test in his specialty. From the given 90 problems he solved 75. To pass the test he needed to solve  $\frac{5}{6}$  of all problems. Did Dmitri pass the test?

# 174. Length of a Circle. Practical Exercises

**Example 1** The frame of a window is in the shape of a rectangle and a semi-circle glued together as shown and has a perimeter of 5.37 m. Find the height of window, if its width is 1 m.



**Solution:**

We denote the rectangular part of the window by  $ABCD$ .

Thus,  $AB = CD = 1$  m.

$$AD = BC = x.$$

$CD$  is the diameter of the semi-circle, and

$$r = \frac{1}{2} \cdot CD = \frac{1}{2} \cdot 1 = 0.5 \text{ m.}$$

The length of the semi-circle is

$$\frac{1}{2} \text{ of } c = \frac{1}{2} \cdot 2 \cdot \pi \cdot r = \pi \cdot r = 3.14 \cdot 0.5 = 1.57 \text{ m.}$$

From the equality  $\frac{1}{2} \cdot c + AB + 2 \cdot x = 5.37$  we find the unknown  $x$ :

$$1.57 + 1 + 2 \cdot x = 5.37$$

$$2.57 + 2 \cdot x = 5.37$$

$$2 \cdot x = 5.37 - 2.57$$

$$2 \cdot x = 2.80$$

$$x = 2.80 \div 2$$

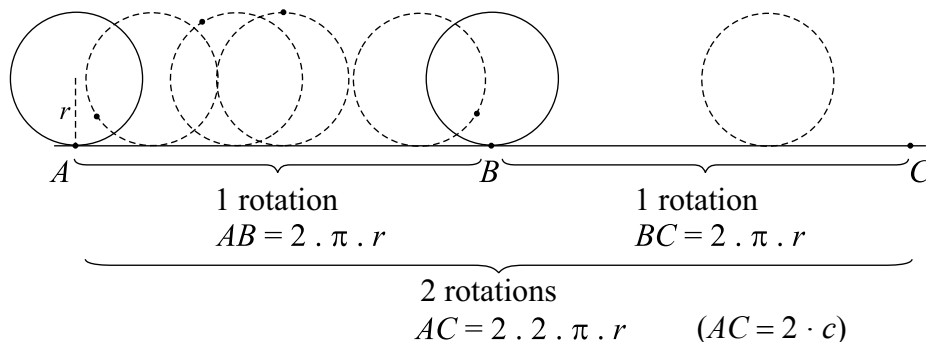
$$x = 1.40.$$

The height of the window is obtained from  $x + r = 1.40 + 0.50 = 1.90$ .

The height of the window is 1.90 m.

**Example 2** A gymnast works with a hoop that has a diameter of 80 cm. How many meters will the hoop travel if it makes 2 full rotations? How about 10 full rotations?

**Solution:**



We obtain the relationship:

**the distance  $s$  = number of rotations  $\cdot$  the length of the circle.**

The length of the circle is thus calculated by:

$$c = 3.14 \cdot 0.8 = 2.512 \text{ m} \quad (80 \text{ cm} = 0.8 \text{ m}).$$

With 2 full rotations  $\rightarrow AC = 2 \cdot c = 2 \cdot 2.512 = 5.024 \text{ m.}$

With 10 full rotations  $\rightarrow 10 \cdot c = 10 \cdot 2.512 = 25.12 \text{ m.}$



**Example 3** A mother is pushing a stroller with her small child. If the front wheel has radius 12 cm, can you find how many times it will have turned after the mother walks 200 meters?

**Solution:**

The wheel has made  $x$  full turns. The distance traveled by it is 200 m. From the relationship **number of turns**  $\cdot c = s$  (the distance traveled)

we find that

$$x \cdot 2 \cdot \pi \cdot r = 20,000 \text{ (in cm)}$$

$$x \cdot 6.28 \cdot 12 = 20,000$$

$$x = 20,000 \div 75.36$$

$$x = 265.39. \text{ The number of full turns is 265.}$$

**Example 4** The tire of a car has radius 40 cm. What is the speed of the car if the tire makes 600 revolutions in one minute?

**Solution:**

The speed of the car is the number of kilometers which it covers in 1 hour (= 60 minutes).

1 revolution = 1 full turn of the wheel; i.e.,

$$1 \text{ revolution} \rightarrow c = 2 \cdot \pi \cdot 40 = 251.2 \text{ cm.}$$

$$\text{In 1 minute it makes 600 revolutions} \rightarrow 600 \cdot 251.2 = 150,720 \text{ cm.}$$

$$\text{In 1 hour it makes } 60 \cdot 600 \text{ revolutions} \rightarrow 60 \cdot 150,720 = 9,043,200 \text{ cm.}$$

$$9,043,200 \text{ cm} = 90,432 \text{ m} = 90.432 \text{ km} \approx 90.43 \text{ km.}$$

In 1 hour the tire covers a distance of 90.43 km; i.e., the speed of the car is  $v = 90.43 \text{ km/h}$ .



*The ancient Babylonians believed that the ratio of the length of every circle and the length of its diameter is the number 3. The Egyptians used the number 3.16.*

*Archimedes (287-212 BCE) determined the number  $\pi$  with accuracy to 2 decimal digits.*

*Ludolph Van Ceulen (1540-1610) introduced the notation  $\pi$  (pi) for this number.*

*Nowadays, with the help of computers,  $\pi$  can be determined up to trillions of decimal digits:*

$$\pi = 3.14159265358979323846 \dots$$

## Exercises

- 1 A round mirror has a diameter of 80 cm. Find out how long its frame is in meters.
- 2 You have a large piece of (flat) glass and you want to cover with it a round table with diameter 1 m. To cut the glass, the handyman asks for \$12, and to smooth out the edge he charges \$1.20 per 10 cm. How many dollars do you have to pay?  
Ani rolls a hoop with a diameter of 80 cm.
- 3 If the hoop makes 5 full turns, how many meters does it travel?
- 4 Milan biked 150 m. How many times did the wheels of the bike turn if both have a diameter of 56 cm?
- 5 The tire of a car has a radius of 40 cm. With what speed does the car travel if the tire makes 550 revolutions in a minute?
- 6 The wheel of a roadtanker has a diameter of 180 cm. In 5 minutes it makes 1,000 revolutions. What is the speed of the roadtanker?