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96. RATIONALIZING IRRATIONAL EXPRESSIONS

EXAMPLE 1 Consider the irrational expressions $A=3+\sqrt{x+2}$ and $B=3-\sqrt{x+2}$, $x\geq-2$.
Perform the operations:

a) A^2; b) B^2; c) $A\cdot B$.

Solution:

a) $A^2=\left(3+\sqrt{x+2}\right)^2$
$=3^2+2\cdot3\sqrt{x+2}+\left(\sqrt{x+2}\right)^2$
$=9+6\sqrt{x+2}+x+2$
$=x+6\sqrt{x+2}+11$

b) $B^2=\left(3-\sqrt{x+2}\right)^2$
$=3^2-2\cdot3\sqrt{x+2}+\left(\sqrt{x+2}\right)^2$
$=9-6\sqrt{x+2}+x+2$
$=x-6\sqrt{x+2}+11$

c) $A\cdot B$
$=\left(3+\sqrt{x+2}\right)\cdot\left(3-\sqrt{x+2}\right)$
$=3^2-\left(\sqrt{x+2}\right)^2$
$=9-(x+2)$
$=7-x$

EXAMPLE 2 Simplify the expression $A=\frac{1}{2}\left(\sqrt{x+1}+\sqrt{x-1}\right)^2$, $x\geq1$, and calculate its numerical value for $x=\sqrt{10}$.

Solution:

1. $A=\frac{1}{2}\left(\sqrt{x+1}+\sqrt{x-1}\right)^2$
$=\frac{1}{2}\left((\sqrt{x+1})^2+2\sqrt{x+1}\sqrt{x-1}+(\sqrt{x-1})^2\right)$
$=\frac{1}{2}(x+1+2\sqrt{x^2-1}+x-1)$
$=\frac{1}{2}(2x+2\sqrt{x^2-1})=x+\sqrt{x^2-1}$

2. For $x=\sqrt{10}$
$A=\sqrt{10}+\sqrt{\left(\sqrt{10}\right)^2-1}$
$=\sqrt{10}+\sqrt{10-1}$
$=3+\sqrt{10}.$

EXAMPLE 3 Simplify the expression $B=\left(\sqrt{x}-\frac{4\sqrt{x}}{2-\sqrt{x}}\right)\cdot\frac{\sqrt{x}-2}{\sqrt{x}+2}$, $x\geq0$, $x\neq4$, and calculate its numerical value for $x=1\frac{9}{16}$.

Solution:

1. $B=\left(\sqrt{x}-\frac{4\sqrt{x}}{2-\sqrt{x}}\right)\cdot\frac{\sqrt{x}-2}{\sqrt{x}+2}$
$=\frac{2\sqrt{x}-x-4\sqrt{x}}{2-\sqrt{x}}\cdot\frac{\sqrt{x}-2}{\sqrt{x}+2}$
$=\frac{\left(-x-2\sqrt{x}\right)\left(\sqrt{x}-2\right)}{\left(2-\sqrt{x}\right)\left(\sqrt{x}+2\right)}$
$=\frac{-x\sqrt{x}+2x-2x+4\sqrt{x}}{4-x}$
$=\frac{\sqrt{x}(4-x)}{4-x}=\sqrt{x}$

2. For $x=1\frac{9}{16}$
$B=\sqrt{\frac{25}{16}}$
$=\frac{\sqrt{25}}{\sqrt{16}}=\frac{5}{4}$
$=1\frac{1}{4}.$

D7 When we rewrite a fraction so that there are no radicals in its denominator, we say that we **rationalize the denominator.**

Example: $\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}}=\frac{2\sqrt{3}}{\left(\sqrt{3}\right)^2}=\frac{2}{3}\sqrt{3}.$

To rationalize a fraction with an irrational denominator, we need to multiply the top and the bottom of the fraction by a *suitably chosen factor* so that the product in the new denominator is a rational expression.

Examples: $\frac{9}{\sqrt{3}}=\frac{9\sqrt{3}}{\sqrt{3}\sqrt{3}}=\frac{9\sqrt{3}}{3}=3\sqrt{3}$,

$$\frac{1}{\sqrt{2}+1}=\frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)}=\frac{\sqrt{2}-1}{(\sqrt{2})^2-1^2}=\frac{\sqrt{2}-1}{2-1}=\sqrt{2}-1.$$

D8 Two irrational expressions whose product is a rational expression are called **conjugate**.

Examples of conjugate expressions: $2-\sqrt{7}$ and $2+\sqrt{7}$; $\sqrt{5}+\sqrt{2}$ and $\sqrt{5}-\sqrt{2}$.

!

Rule 6: $\frac{a}{\sqrt{b}}=\frac{a\sqrt{b}}{\sqrt{b}\sqrt{b}}=\frac{a\sqrt{b}}{b},\quad b>0$

$$\frac{a}{\sqrt{b}\pm\sqrt{c}}=\frac{a(\sqrt{b}\mp\sqrt{c})}{(\sqrt{b}\pm\sqrt{c})(\sqrt{b}\mp\sqrt{c})}=\frac{a(\sqrt{b}\mp\sqrt{c})}{b-c},\quad b\ge 0,\ c\ge 0,\ b\ne c.$$

EXAMPLE 4 Rationalize the denominator of the fraction:

a) $\frac{5}{3\sqrt{2}}$; b) $\frac{4}{\sqrt{7}-\sqrt{5}}$; c) $\frac{12}{3-\sqrt{5}}$.

Solution:

a) $\frac{5}{3\sqrt{2}}=\frac{5\sqrt{2}}{3\sqrt{2}\sqrt{2}}=\frac{5\sqrt{2}}{3\cdot 2}=\frac{5\sqrt{2}}{6}$

b) $\frac{4}{\sqrt{7}-\sqrt{5}}=\frac{4\cdot(\sqrt{7}+\sqrt{5})}{(\sqrt{7}-\sqrt{5})(\sqrt{7}+\sqrt{5})}=\frac{4\cdot(\sqrt{7}+\sqrt{5})}{7-5}=2\cdot(\sqrt{7}+\sqrt{5})$

c) $\frac{12}{3-\sqrt{5}}=\frac{12\cdot(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})}=\frac{12\cdot(3+\sqrt{5})}{9-5}=3\cdot(3+\sqrt{5})$

EXAMPLE 5 Rationalize the denominator of the expression:

a) $\frac{2a+1}{\sqrt{a}}$, $a\ge 0$;

b) $\frac{8}{\sqrt{x}+\sqrt{x-2}}$, $x\ge 2$;

c) $\frac{3x}{2+\sqrt{x+4}}$, $x\ge 4$, $x\ne 0$.

Solution: a) $\frac{2a+1}{\sqrt{a}}=\frac{(2a+1)\sqrt{a}}{\sqrt{a}\cdot\sqrt{a}}=\frac{(2a+1)\sqrt{a}}{a}$.

b) $\frac{8}{\sqrt{x}+\sqrt{x-2}}=\frac{8(\sqrt{x}-\sqrt{x-2})}{(\sqrt{x}+\sqrt{x-2})(\sqrt{x}-\sqrt{x-2})}=\frac{8(\sqrt{x}-\sqrt{x-2})}{(\sqrt{x})^2-(\sqrt{x-2})^2}=\frac{8(\sqrt{x}-\sqrt{x-2})}{x-(x-2)}=4(\sqrt{x}-\sqrt{x-2})$

c) $\frac{3x}{2+\sqrt{x+4}}=\frac{3x(2-\sqrt{x+4})}{(2+\sqrt{x+4})(2-\sqrt{x+4})}=\frac{3x(2-\sqrt{x+4})}{2^2-(\sqrt{x+4})^2}=\frac{3x(2-\sqrt{x+4})}{4-(x+4)}=3(\sqrt{x+4}-2)$

EXERCISES

1. Perform the operations below for the expressions: $A=x+\sqrt{x^2+7}$ and $B=x-\sqrt{x^2+7}$.

 a) A^2; b) B^2; c) $A\cdot B$.

2. Simplify $C=\frac{1}{2}(\sqrt{x+2}-\sqrt{x-2})^2$, $x\ge 2$, and calculate its value for $x=2\sqrt{5}$.

112. PROPERTIES OF ARITHMETIC PROGRESSIONS

EXAMPLE 1 Consider the arithmetic progression (÷) 5, 8, 11, 14, 17, ... We notice that:

- $8 = \frac{5+11}{2}$, $11 = \frac{8+14}{2}$, $14 = \frac{11+17}{2}$;

i.e., every term after the first is the arithmetic average of its two neighbors.

T2

Property 2: A sequence $a_1, a_2, a_3, ..., a_{n-1}, a_n, a_{n+1}, ...$ is an arithmetic progression if and only if $\boldsymbol{a_n = \frac{a_{n-1} + a_{n+1}}{2}}$ for all $n \geq 2$.

Proof:

- If a_{n-1}, a_n, and a_{n+1} $(n \geq 2)$ are three consecutive terms of an arithmetic progression, from the definition, it follows that

$$d = a_n - a_{n-1} = a_{n+1} - a_n, \quad 2a_n = a_{n-1} + a_{n+1}, \quad a_n = \frac{a_{n-1} + a_{n+1}}{2}.$$

- Conversely, if for any three consecutive terms a_{n-1}, a_n, and a_{n+1} of a sequence the equality $a_n = \frac{a_{n-1} + a_{n+1}}{2}$ is true, it follows that

$$2a_n = a_{n-1} + a_{n+1}; \text{ i.e., } a_n - a_{n-1} = a_{n+1} - a_n \text{ for any } n \geq 2.$$

From here, for $n = 2, 3, 4, ...$, we obtain:

$$\left| \begin{array}{l} a_2 - a_1 = a_3 - a_2 \\ a_3 - a_2 = a_4 - a_3 \\ a_4 - a_3 = a_5 - a_4 \\ \text{etc.,} \end{array} \right. \Rightarrow a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = a_5 - a_4 = ...,$$

which shows that the sequence is arithmetic progression.

EXAMPLE 2 For which values of x are the numbers 3, $x + 4$, and $x^2 + 2$ consecutive terms of an increasing arithmetic progression?

Solution: From *Property 2* of arithmetic progressions, we have:

$$a_n = \frac{a_{n-1} + a_{n+1}}{2} \qquad x + 4 = \frac{3 + x^2 + 2}{2}$$

$$a_{n-1} = 3 \qquad 2x + 8 = x^2 + 5$$

$$a_n = x + 4 \qquad x^2 - 2x - 3 = 0$$

$$a_{n+1} = x^2 + 2 \qquad x_1 = 3, \quad x_2 = -1,$$

(÷) 3, 7, 11; (÷) 3, 3, 3.

Answer: For $x = 3$, the numbers 3, $x + 4$, and $x^2 + 2$ are consecutive terms of an increasing arithmetic progression (with $d = 4$).

EXAMPLE 3 For what x are 8, $2x + 1$, and $x^2 - 2x - 1$ consecutive terms of a decreasing arithmetic progression?

Solution:

$$a_n = \frac{a_{n-1} + a_{n+1}}{2} \qquad 2x + 1 = \frac{8 + x^2 - 2x - 1}{2}$$

$$a_{n-1} = 8 \qquad 4x + 2 = x^2 - 2x + 7$$

$$a_n = 2x + 1 \qquad x^2 - 6x + 5 = 0$$

$$a_{n+1} = x^2 - 2x - 1 \qquad x_1 = 1, \quad x_2 = 5,$$

(÷) 8, 3, –2; (÷) 8, 11, 14.

Answer: For $x = 1$, the numbers 8, $2x + 1$, and $x^2 - 2x - 1$ are consecutive terms of a decreasing arithmetic progression (with $d = -5$).

EXAMPLE 4 Consider the arithmetic progression

$$(\div)\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10.$$

We notice that $1 + 10 = 2 + 9 = 3 + 8 = 4 + 7 = 5 + 6\ \ (= 11)$.

This property works for the terms of *any* finite arithmetic progression.

T3

Property 3: In any finite arithmetic progression (÷) $a_1, a_2, a_3, \ldots, a_{n-2}, a_{n-1}, a_n$, the sum of two terms that are at equal distances from the two end terms is equal to the sum of the end terms; i.e., $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots.$

We will generalize *Property 3*.

For any arithmetic progression (finite or infinite) the sum of any two terms satisfies

$$a_p + a_q = a_1 + (p-1)d + a_1 + (q-1)d = 2a_1 + ((p+q)-2)d,$$

$$a_r + a_s = a_1 + (r-1)d + a_1 + (s-1)d = 2a_1 + ((r+s)-2)d$$

and hence this sum depends only on the sum of the terms' indices. Therefore,

T3'

Property 3': If two pairs of indices add up to the same sum, then the corresponding pairs of terms of an arithmetic progression also add up to the same sum:

$$p + q = r + s \ \Rightarrow\ a_p + a_q = a_r + a_s.$$

In particular, the sums $a_1 + a_n$, $a_2 + a_{n-1}$, $a_3 + a_{n-2}$, ... are all equal since the corresponding indices also add up to equal sums: $1 + n = 2 + (n-1) = 3 + (n-2) = \ldots.$

EXAMPLE 5 For an arithmetic progression, we know that $a_4 + a_8 = 42$. Find:

a) a_6;

b) $a_3 + a_9$.

Solution: From *Property 3* of arithmetic progressions, we have:

a) $a_4 + a_8 = a_6 + a_6\ (4 + 8 = 6 + 6)$

$42 = 2a_6$

$a_6 = 21$;

b) $a_4 + a_8 = a_3 + a_9\ (4 + 8 = 3 + 9)$

$42 = a_3 + a_9$

$a_3 + a_9 = 42.$

EXAMPLE 6 For an arithmetic progression, we know that $a_8 = 12$. Find:

a) $a_7 + a_9$;

b) $a_2 + a_{14}$.

Solution:

a) $a_8 + a_8 = a_7 + a_9\ (8 + 8 = 7 + 9)$

$12 + 12 = a_7 + a_9$

$a_7 + a_9 = 24$;

b) $a_8 + a_8 = a_2 + a_{14}\ (8 + 8 = 2 + 14)$

$12 + 12 = a_2 + a_{14}$

$a_2 + a_{14} = 24.$

EXERCISES

For which value of x are the numbers:

1. $x - 1, x + 3$, and $x^2 - 5$ positive consecutive terms of an arithmetic progression?
2. $x - 2$, $x + 4$, and $x^2 - 2$ consecutive terms of an increasing arithmetic progression?
3. 5, $x + 5$, and $x^2 + 2$ consecutive terms of a decreasing arithmetic progression?

In an arithmetic progression, we know that:

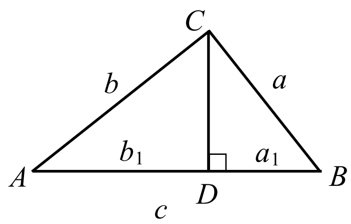
4. $a_7 + a_{13} = 48$. Find: a) a_{10}; b) $a_5 + a_{15}$.
5. $a_2 + a_{10} = 24$.

   Find: a) $a_5 + a_6 + a_7$; b) $a_4 + a_5 + a_6 + a_7 + a_8$.
6. $a_{12} = 64$. Find: a) $a_{10} + a_{14}$; b) $a_6 + a_{18}$.

139. THE PYTHAGOREAN THEOREM

T3 **Pythagorean Theorem**
The sum of the squares of the legs in a right triangle is equal to the square of the hypotenuse: $a^2 + b^2 = c^2$.

Proof:



We know that $\left|\begin{aligned} a^2 &= a_1 \cdot c \\ b^2 &= b_1 \cdot c \end{aligned}\right\} +$

We add the two equalities and obtain:

$$a^2 + b^2 = a_1 \cdot c + b_1 \cdot c =$$
$$= c \cdot (a_1 + b_1) = c \cdot c = c^2.$$

The **converse theorem** is also true:

T4 If the sum of the squares of two sides is in a triangle is equal to the square of he third side, then the triangle is right.

The two theorems can be formulated as one in the following way:

T5 ΔABC with sides $BC = a$, $CA = b$, and $AB = c$ is right with a right angle at vertex C if and only if $a^2 + b^2 = c^2$:

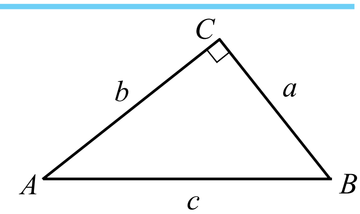
$\Delta ABC, \angle C = 90° \Leftrightarrow a^2 + b^2 = c^2$.

EXAMPLE 1 Find the third side of ΔABC ($\angle C = 90°$) with legs a and b and hypotenuse c, provided:

a) $a = 6$ and $b = 8$;

b) $a = 5$ and $c = 13$;

c) $b = \sqrt{7}$ and $c = 4$.



Solution:

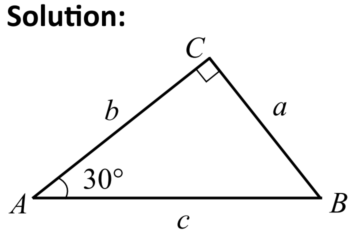
a)
$$a^2 + b^2 = c^2$$
$$6^2 + 8^2 = c^2$$
$$36 + 64 = c^2$$
$$c^2 = 100$$
$$c = 10$$

b)
$$a^2 + b^2 = c^2$$
$$5^2 + b^2 = 13^2$$
$$25 + b^2 = 169$$
$$b^2 = 144$$
$$b = 12$$

c)
$$a^2 + b^2 = c^2$$
$$a^2 + (\sqrt{7})^2 = 4^2$$
$$a^2 + 7 = 16$$
$$a^2 = 9$$
$$a = 3$$

EXAMPLE 2 Find the legs of a right ΔABC ($\angle C = 90°$) with $\angle BAC = 30°$ and $AB = 8$ cm.

Solution:



1. $a = \frac{c}{2} = \frac{8}{2} = 4$ cm (leg opposite 30°)

2. $a^2 + b^2 = c^2 \Rightarrow 4^2 + b^2 = 8^2 \Rightarrow$

$b^2 = 8^2 - 4^2 = 48 \Rightarrow b = \sqrt{48} = 4\sqrt{3}$ cm

EXAMPLE 3 Is ΔABC right, provided its sides are:

a) $a=\sqrt{5}, b=2, c=3;$ b) $a=3, b=4, c=5;$ c) $a=\sqrt{17}, b=2\sqrt{2}, c=5?$

Solution:

a) $a^2+b^2=$
$=(\sqrt{5})^2+2^2=$
$=5+4=9$
$c^2=3^2=9$
$\Rightarrow a^2+b^2=c^2$
$\Rightarrow \angle C=90°$

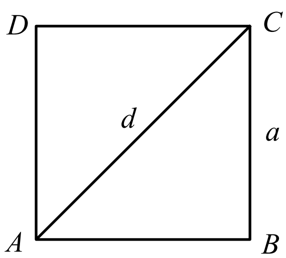
b) $a^2+b^2=$
$=3^2+4^2=$
$=9+16=25$
$c^2=5^2=25$
$\Rightarrow a^2+b^2=c^2$
$\Rightarrow \angle C=90°$

c) $a^2+b^2=$
$=\left(\sqrt{17}\right)^2+\left(2\sqrt{2}\right)^2=$
$=17+8=25$
$c^2=5^2=25$
$\Rightarrow a^2+b^2=c^2$
$\Rightarrow \angle C=90°$

Answer: Yes, ΔABC is right.

EXAMPLE 4 Find the diagonal of a square with side a.

Solution:

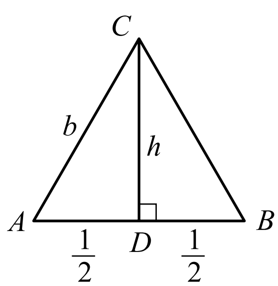


We apply the Pythagorean Theorem to ΔABC:

$AC^2=AB^2+BC^2$
$d^2=a^2+a^2$
$d^2=2a^2$
$d=a\sqrt{2}$

EXAMPLE 5 Find the altitude of an equilateral triangle with side b.

Solution:



1. $AD=BD=\frac{b}{2}$ (CD altitude, median)

2. We apply the Pythagorean Theorem to ΔACD:

$AC^2=AD^2+CD^2$

$b^2=\left(\frac{b}{2}\right)^2+h^2$

$h^2=b^2-\frac{b^2}{4}\Rightarrow h^2=\frac{3b^2}{4}\Rightarrow h=\frac{b\sqrt{3}}{2}$

EXERCISES

1. Find the third side of ΔABC ($\angle C=90°$) with legs a and b and hypotenuse c if:

a) $a=1$ and $b=1$;

b) $a=2$ and $c=5$;

c) $a=5$ and $b=12$;

d) $b=5$ and $c=13$.

2. Find the legs of ΔABC with $\angle C=90°$, $\angle BAC=60°$, and $AB=10$ cm.

3. Find the legs of an isosceles right triangle with hypotenuse 4 cm.

4. Check if the triangle is right, provided its sides have the following lengths:

a) $a=15, b=20$, and $c=25$;

b) $a=4,\ b=8,$ and $c=4\sqrt{5}$;

c) $a=7,\ b=24$, and $c=25$;

d) $a=6,\ b=8,$ and $c=12$.