

9B Contents

Icons, Symbols, Notation	II
Foreword.....	III
Reviews of the Program	IV
Contents.....	VIII
Introduction to the Program.....	X
1. Mission Impossible. 2. The Find.....	X
3. Origins. 4. Purpose.....	XI
5. The Temple of Mathematics	XII
6. Relationship with Mathematics	XIV
7. A Cultural Shift.....	XV
8. The Pillars of Mathematics	XVI
8.1. Algebra. 8.2. The Missing Algebraic Link	XVI
8.3. The Geometry Neverland.....	XVIII
8.4. Golden Geometry Days of My Childhood....	XIX
8.5. A Reality Check: Geo-Algebra in the U.S.?....	XX
Math Curriculum at a Glance.....	XX-XXI
8.6. Bridge From High School to College	XXI
8.7-8.8. Structure of All Topics.....	XXII-XXIII
8.8. The Main Principles.....	XXIV
9. Differentiate and Scaffold	XXIV
9.1. Examples of Scaffolding.....	XXV
9.2. What are Optional Topics?	XXV
9.3. Does Scaffolding Really Work?	XXV
10. The Value of a Teacher.....	XXVI
11. Can a School Succeed w/ the Program?	XXVIII
11.1. The U.S. Students	XXVIII
11.2. Can Girls Succeed in Math?.....	XXVIII
11.3. The Needed Cultural Shift	XXIX
11.4. Continuity of the Curriculum.....	XXIX
11.5. Local Support	XXIX
12. Within the Global U.S. Picture	XXX
13. How to Work with Math 5/9A Materials.....	XXXI
14. Biographical Sketches	XXXI
15. Math Taught the Right Way	XXXIV
16. BMC Summer Program	XXXV
17. Acknowledgments	XXXVI

Let the Math Begin!

Chapter 8. Irrational Expressions and Equations

93. Operations with Square Roots (Review) ...	186
94. Working with Square Roots (Review)	188
95. Irrational Expressions (Review)	190
96. Rationalizing Irrational Expressions.....	192
97. Rationalizing Irrational Expressions. Exercises	194
98. Irrational Equations with One Radical.....	196
99. Irrational Equations with Two Radicals.....	198

100. Irrational Equations. Exercises	200
101. Domains in Irrational Equations. Exercises	202
102. Introducing New Variables in Irrational Equations.....	204
103. Introducing New Variables in Irrational Equations. Exercises	206
104. Equivalence of Irrational Equations.....	208
105. Summary of “Irrational Expressions and Equations,” I	210
106. Summary of “Irrational Expressions and Equations,” II	212
107. Tests on “Irrational Expressions/Equations”	214
108. General Problems on “Irrational Expressions and Equations”	216

Chapter 9. Progressions

109. Introduction to Sequences. Ways to Define Sequences	218
110. Recurrence and Monotone Sequences	220
111. Arithmetic Progressions. Formula for Their General Term.....	222
112. Properties of Arithmetic Progressions.....	224
113. Partial Sum of Arithmetic Progressions	226
114. Geometric Progressions. Formula for Their General Term.....	228
115. Properties of Geometric Progressions	230
116. Partial Sum of Geometric Progressions	232
117. Combined Exercises on Progressions, I....	234
118. Combined Exercises on Progressions, II....	236
119. Interest. Simple Interest	238
120. Compound Interest	240
121. Practical Problems with Interest. Financial Products: Credit, Rent, and Lease..	242
122. Summary of “Progressions,” I	244
123. Summary of “Progressions,” II	246
124. Tests on “Progressions”	248
125. General Problems on “Progressions”	250

Chapter 10. Introduction to Statistics

126. Descriptive Statistics. Populations and Samples	252
127. Data Processing and Presentation	254
128. Measures of Central Tendency - Arithmetic Mean and Median	256
129. Measure of Central Tendency - Mode. Exercises	258
130. The 5-Number Summary of Data.....	260
131. Box-and-Whisker Plots	262

Theory Overview and Practical Applications

- 132. Statistics as a Science, I 264
- 133. Populations and Samples, II 266
- 134. Series, *M*-Measures, III 268
- 135. 5-Number Summary. Bernoulli Trials, IV ... 270
- 136. Simpson's Paradox. More on Averages, V ... 272
- 137. Tests on “Statistics and Data Processing” .. 274

Chapter 11. Metric Relations btw Segments

- 138. Metric Relations between Segments
in a Right Triangle..... 276
- 139. The Pythagorean Theorem 278
- 140. The Pythagorean Theorem. Exercises..... 280
- 141. Length of a Segment: Distance Formula ... 282
- 142. Solving a Right Triangle 284
- 143. Applying Metric Relations to Solving
a Right Triangle. Exercises 286
- 144. Solving an Isosceles Triangle..... 288
- 145. Solving Isosceles Triangle. Exercises 290
- 146. Solving Isosceles and Right Trapezoids..... 292
- 147. Solving a Parallelogram..... 294
- 148. Metric Relations btw Segments in a Circle 296
- 149. Metric Relations btw Segments in a Circle.
Exercises 298
- 150. Summary of “Metric Relations” 300
- 151. Tests on “Metric Relations btw Segments” .. 302
- 152. General Problems on “Metric Relations” 304

Chapter 12. Introduction to Trigonometry Trigonometric Functions for an Acute Angle

- 153. Trigonometric Functions for an Acute Angle 306
- 154. Values of the Basic Trigonometric Functions
for 30° , 45° , and 60° Angles 308
- 155. Basic Relations btw Trigonometric Functions
for an Acute Angle 310
- 156. Trigonometric Functions
for an Acute Angle. Exercises..... 312
- 157. Trigonometric Functions
for a Complementary Acute Angle 314
- 158. Solving a Right Triangle 316
- 159. Solving a Right Triangle. Exercises..... 318
- 160. Solving an Isosceles Triangle..... 320
- 161. Solving an Isosceles and Right Trapezoid... 322
- 162. Applying Trigonometric Functions
for an Acute Angle. Practical Problems 324
- 163. Summary of “Trigonometry Intro” 326
- 164. Tests on “Trigonometry Intro” 328
- 165. General Problems on “Trigonometry Intro” 330

Chapter 13 Trigonometry in the Plane

Solving a Triangle

- 166. Trigonometric Functions on $[0^\circ, 180^\circ]$ 332
- 167. Trigonometric Identities on $[0^\circ, 180^\circ]$ 334
- 168. Trig. Values for Some Angles in $[0^\circ, 180^\circ]$ 336
- 169. Calculating Trig. Expressions. Exercises... 338
- 170. The Law of Sines 340
- 171. Solving Any Triangle: Basic Examples 342
- 172. Solving Any Triangle w/ the Law of Sines 344
- 173. The Law of Cosines 346
- 174. Solving Any Triangle w/ the Law of Cosines 348
- 175. Medians in a Triangle 350
- 176. Angle Bisectors in a Triangle..... 352
Formulas for the Area of a Triangle:
- 177. Via Inradius or Sides and an Angle, I..... 354
- 178. Heron's Formula or via Circumradius, II ... 356
- 179. Area of a Triangle. Exercises, III 358
- 180. Summary of “Trigonometry in the Plane” . 360
- 181. Tests on “Trigonometry in the Plane” 362
- 182. General Problems on “Solving a Triangle” 364

Chapter 14. Yearly Review

- 183. Quadratic Equations. Vieta's Formulas 366
- 184. Rational and Irrational Equations 368
- 185. Systems of Linear Equations and Beyond . 370
- 186. Systems of Degree 2 Equations 372
- 187. Inequalities: Linear, Modular, Quadratic ... 374
- 188. The Method of the Intervals..... 376
- 189. Linear Functions and Graphs 378
- 190. Quadratic Functions. Optimization..... 380
- 191. Sequences. Arithmetic Progressions 382
- 192. Geometric Progressions. Interest Rates 384
- 193. Similar Triangles. Pythagorean Theorem ... 386
- 194. Metric Relations btw Segments in Circles. 388
- 195. Trigonometric Functions. Basic Identities . 390
- 196. Trigonometry Problems in a Right Triangle 392
Solving a Triangle:
- 197. Part I: Laws of Sines and Cosines..... 394
- 198. Part II: Special Elements. Area 396
- 199. Basic Combinatorial Concepts..... 398
- 200. Classical Probabilities 400
- 201. Statistics and Data Processing 402
- 202. Combinatorics, Probability, Statistics 404
- 203-204. Preparation for Exit Tests 1-2 406
- 205-206. Exit Level Tests 1-2 with Solutions.... 410
- 207. Exit Level Tests 1-2 414

Answers.....XXXVII

Summary of Concepts and StatementsXLI

Appendix.....XLVII

EXAMPLE 1 Consider the irrational expressions $A=3+\sqrt{x+2}$ and $B=3-\sqrt{x+2}$, $x \geq -2$. Perform the operations:

a) A^2 ; b) B^2 ; c) $A \cdot B$.

Solution:

$$\begin{array}{lll} \text{a) } A^2 = (3+\sqrt{x+2})^2 & \text{b) } B^2 = (3-\sqrt{x+2})^2 & \text{c) } A \cdot B \\ = 3^2 + 2 \cdot 3\sqrt{x+2} + (\sqrt{x+2})^2 & = 3^2 - 2 \cdot 3\sqrt{x+2} + (\sqrt{x+2})^2 & = (3+\sqrt{x+2}) \cdot (3-\sqrt{x+2}) \\ = 9 + 6\sqrt{x+2} + x + 2 & = 9 - 6\sqrt{x+2} + x + 2 & = 3^2 - (\sqrt{x+2})^2 \\ = x + 6\sqrt{x+2} + 11 & = x - 6\sqrt{x+2} + 11 & = 9 - (x+2) \\ & & = 7 - x \end{array}$$

EXAMPLE 2 Simplify the expression $A = \frac{1}{2}(\sqrt{x+1} + \sqrt{x-1})^2$, $x \geq 1$, and calculate its numerical value for $x = \sqrt{10}$.

Solution:

$$\begin{array}{ll} \text{1. } A = \frac{1}{2}(\sqrt{x+1} + \sqrt{x-1})^2 & \text{2. For } x = \sqrt{10} \\ = \frac{1}{2}((\sqrt{x+1})^2 + 2\sqrt{x+1}\sqrt{x-1} + (\sqrt{x-1})^2) & A = \sqrt{10} + \sqrt{(\sqrt{10})^2 - 1} \\ = \frac{1}{2}(x+1 + 2\sqrt{x^2-1} + x-1) & = \sqrt{10} + \sqrt{10-1} \\ = \frac{1}{2}(2x + 2\sqrt{x^2-1}) = x + \sqrt{x^2-1} & = 3 + \sqrt{10}. \end{array}$$

EXAMPLE 3 Simplify the expression $B = \left(\sqrt{x} - \frac{4\sqrt{x}}{2-\sqrt{x}} \right) \cdot \frac{\sqrt{x}-2}{\sqrt{x}+2}$, $x \geq 0$, $x \neq 4$, and calculate its numerical value for $x = 1\frac{9}{16}$.

Solution:

$$\begin{array}{ll} \text{1. } B = \left(\sqrt{x} - \frac{4\sqrt{x}}{2-\sqrt{x}} \right) \cdot \frac{\sqrt{x}-2}{\sqrt{x}+2} & \text{2. For } x = 1\frac{9}{16} \\ = \frac{2\sqrt{x} - x - 4\sqrt{x}}{2-\sqrt{x}} \cdot \frac{\sqrt{x}-2}{\sqrt{x}+2} & B = \sqrt{\frac{25}{16}} \\ = \frac{(-x-2\sqrt{x})(\sqrt{x}-2)}{(2-\sqrt{x})(\sqrt{x}+2)} & = \frac{\sqrt{25}}{\sqrt{16}} = \frac{5}{4} \\ = \frac{-x\sqrt{x} + 2x - 2x + 4\sqrt{x}}{4-x} & = 1\frac{1}{4}. \\ = \frac{\sqrt{x}(4-x)}{4-x} = \sqrt{x} & \end{array}$$

D7

When we rewrite a fraction so that there are no radicals in its denominator, we say that we **rationalize the denominator**.

Example: $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{2\sqrt{3}}{(\sqrt{3})^2} = \frac{2}{3}\sqrt{3}$.



To rationalize a fraction with an irrational denominator, we need to multiply the top and the bottom of the fraction by a *suitably chosen factor* so that the product in the new denominator is a rational expression.

Examples: $\frac{9}{\sqrt{3}} = \frac{9\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{9\sqrt{3}}{3} = 3\sqrt{3}$,

$$\frac{1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)} = \frac{\sqrt{2}-1}{(\sqrt{2})^2-1^2} = \frac{\sqrt{2}-1}{2-1} = \sqrt{2}-1.$$

D8

Two irrational expressions whose product is a rational expression are called **conjugate**.

Examples of conjugate expressions: $2-\sqrt{7}$ and $2+\sqrt{7}$; $\sqrt{5}+\sqrt{2}$ and $\sqrt{5}-\sqrt{2}$.



Rule 6: $\frac{a}{\sqrt{b}} = \frac{a\sqrt{b}}{\sqrt{b}\sqrt{b}} = \frac{a\sqrt{b}}{b}$, $b > 0$

$$\frac{a}{\sqrt{b} \pm \sqrt{c}} = \frac{a(\sqrt{b} \mp \sqrt{c})}{(\sqrt{b} \pm \sqrt{c})(\sqrt{b} \mp \sqrt{c})} = \frac{a(\sqrt{b} \mp \sqrt{c})}{b-c}, \quad b \geq 0, c \geq 0, b \neq c.$$

EXAMPLE 4 Rationalize the denominator of the fraction:

a) $\frac{5}{3\sqrt{2}}$;

b) $\frac{4}{\sqrt{7}-\sqrt{5}}$;

c) $\frac{12}{3-\sqrt{5}}$.

Solution:

$$\begin{aligned} \text{a) } \frac{5}{3\sqrt{2}} &= \frac{5\sqrt{2}}{3\sqrt{2}\sqrt{2}} \\ &= \frac{5\sqrt{2}}{3 \cdot 2} = \frac{5\sqrt{2}}{6} \end{aligned}$$

$$\begin{aligned} \text{b) } \frac{4}{\sqrt{7}-\sqrt{5}} &= \frac{4 \cdot (\sqrt{7}+\sqrt{5})}{(\sqrt{7}-\sqrt{5})(\sqrt{7}+\sqrt{5})} \\ &= \frac{4 \cdot (\sqrt{7}+\sqrt{5})}{7-5} \\ &= 2 \cdot (\sqrt{7}+\sqrt{5}) \end{aligned}$$

$$\begin{aligned} \text{c) } \frac{12}{3-\sqrt{5}} &= \frac{12 \cdot (3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} \\ &= \frac{12 \cdot (3+\sqrt{5})}{9-5} \\ &= 3 \cdot (3+\sqrt{5}) \end{aligned}$$

EXAMPLE 5 Rationalize the denominator of the expression:

a) $\frac{2a+1}{\sqrt{a}}$, $a \geq 0$;

b) $\frac{8}{\sqrt{x}+\sqrt{x-2}}$,
 $x \geq 2$;

c) $\frac{3x}{2+\sqrt{x+4}}$,
 $x \geq 4, x \neq 0$.

Solution: a) $\frac{2a+1}{\sqrt{a}} = \frac{(2a+1)\sqrt{a}}{\sqrt{a}\sqrt{a}} = \frac{(2a+1)\sqrt{a}}{a}$.

$$\begin{aligned} \text{b) } \frac{8}{\sqrt{x}+\sqrt{x-2}} &= \frac{8(\sqrt{x}-\sqrt{x-2})}{(\sqrt{x}+\sqrt{x-2})(\sqrt{x}-\sqrt{x-2})} \\ &= \frac{8(\sqrt{x}-\sqrt{x-2})}{(\sqrt{x})^2 - (\sqrt{x-2})^2} \\ &= \frac{8(\sqrt{x}-\sqrt{x-2})}{x - (x-2)} \\ &= 4(\sqrt{x}-\sqrt{x-2}) \end{aligned}$$

$$\begin{aligned} \text{c) } \frac{3x}{2+\sqrt{x+4}} &= \frac{3x(2-\sqrt{x+4})}{(2+\sqrt{x+4})(2-\sqrt{x+4})} \\ &= \frac{3x(2-\sqrt{x+4})}{2^2 - (\sqrt{x+4})^2} \\ &= \frac{3x(2-\sqrt{x+4})}{4 - (x+4)} \\ &= 3(\sqrt{x+4}-2) \end{aligned}$$

EXERCISES 1. Perform the operations below for the expressions: $A = x + \sqrt{x^2+7}$ and $B = x - \sqrt{x^2+7}$.

a) A^2 ; b) B^2 ; c) $A \cdot B$.

2. Simplify $C = \frac{1}{2}(\sqrt{x+2} - \sqrt{x-2})^2$, $x \geq 2$, and calculate its value for $x = 2\sqrt{5}$.

EXAMPLE 1 Consider the arithmetic progression (+) 5, 8, 11, 14, 17, We notice that:

$$\bullet \quad 8 = \frac{5+11}{2}, \quad 11 = \frac{8+14}{2}, \quad 14 = \frac{11+17}{2};$$

i.e., every term after the first is the arithmetic average of its two neighbors.

T₂

Property 2: A sequence $a_1, a_2, a_3, \dots, a_{n-1}, a_n, a_{n+1}, \dots$ is an arithmetic progression if and only if $a_n = \frac{a_{n-1} + a_{n+1}}{2}$ for all $n \geq 2$.

Proof:

- If a_{n-1} , a_n , and a_{n+1} ($n \geq 2$) are three consecutive terms of an arithmetic progression, from the definition, it follows that

$$d = a_n - a_{n-1} = a_{n+1} - a_n, \quad 2a_n = a_{n-1} + a_{n+1}, \quad a_n = \frac{a_{n-1} + a_{n+1}}{2}.$$

- Conversely, if for any three consecutive terms a_{n-1} , a_n , and a_{n+1} of a sequence the equality $a_n = \frac{a_{n-1} + a_{n+1}}{2}$ is true, it follows that

$$2a_n = a_{n-1} + a_{n+1}; \text{ i.e., } a_n - a_{n-1} = a_{n+1} - a_n \text{ for any } n \geq 2.$$

From here, for $n = 2, 3, 4, \dots$, we obtain:

$$\left. \begin{array}{l} a_2 - a_1 = a_3 - a_2 \\ a_3 - a_2 = a_4 - a_3 \\ a_4 - a_3 = a_5 - a_4 \\ \text{etc.,} \end{array} \right\} \Rightarrow a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = a_5 - a_4 = \dots,$$

which shows that the sequence is arithmetic progression.

EXAMPLE 2 For which values of x are the numbers 3, $x + 4$, and $x^2 + 2$ consecutive terms of an increasing arithmetic progression?

Solution: From *Property 2* of arithmetic progressions, we have:

$$\begin{array}{ll} a_n = \frac{a_{n-1} + a_{n+1}}{2} & x + 4 = \frac{3 + x^2 + 2}{2} \\ a_{n-1} = 3 & 2x + 8 = x^2 + 5 \\ a_n = x + 4 & x^2 - 2x - 3 = 0 \\ a_{n+1} = x^2 + 2 & x_1 = 3, \quad x_2 = -1, \\ & (+) 3, 7, 11; \quad (+) 3, 3, 3. \end{array}$$

Answer: For $x = 3$, the numbers 3, $x + 4$, and $x^2 + 2$ are consecutive terms of an increasing arithmetic progression (with $d = 4$).

EXAMPLE 3 For what x are 8, $2x + 1$, and $x^2 - 2x - 1$ consecutive terms of a decreasing arithmetic progression?

Solution:

$$\begin{array}{ll} a_n = \frac{a_{n-1} + a_{n+1}}{2} & 2x + 1 = \frac{8 + x^2 - 2x - 1}{2} \\ a_{n-1} = 8 & 4x + 2 = x^2 - 2x + 7 \\ a_n = 2x + 1 & x^2 - 6x + 5 = 0 \\ a_{n+1} = x^2 - 2x - 1 & x_1 = 1, \quad x_2 = 5, \\ & (+) 8, 3, -2; \quad (+) 8, 11, 14. \end{array}$$

Answer: For $x = 1$, the numbers 8, $2x + 1$, and $x^2 - 2x - 1$ are consecutive terms of a decreasing arithmetic progression (with $d = -5$).

EXAMPLE 4 Consider the arithmetic progression

$$(+)\ 1, \ 2, \ 3, \ 4, \ 5, \ 6, \ 7, \ 8, \ 9, \ 10.$$

We notice that $1 + 10 = 2 + 9 = 3 + 8 = 4 + 7 = 5 + 6 (= 11)$.

This property works for the terms of *any* finite arithmetic progression.

T₃

Property 3: In any finite arithmetic progression $(+)$ $a_1, a_2, a_3, \dots, a_{n-2}, a_{n-1}, a_n$, the sum of two terms that are at equal distances from the two end terms is equal to the sum of the end terms; i.e., $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \dots$

We will generalize *Property 3*.

For any arithmetic progression (finite or infinite) the sum of any two terms satisfies

$$a_p + a_q = a_1 + (p-1)d + a_1 + (q-1)d = 2a_1 + ((p+q)-2)d,$$

$$a_r + a_s = a_1 + (r-1)d + a_1 + (s-1)d = 2a_1 + ((r+s)-2)d$$

and hence this sum depends only on the sum of the terms' indices. Therefore,

T'₃

Property 3': If two pairs of indices add up to the same sum, then the corresponding pairs of terms of an arithmetic progression also add up to the same sum:

$$p + q = r + s \Rightarrow a_p + a_q = a_r + a_s.$$

In particular, the sums $a_1 + a_n, a_2 + a_{n-1}, a_3 + a_{n-2}, \dots$ are all equal since the corresponding indices also add up to equal sums: $1 + n = 2 + (n-1) = 3 + (n-2) = \dots$

EXAMPLE 5 For an arithmetic progression, we know that $a_4 + a_8 = 42$. Find:

a) a_6 ;

b) $a_3 + a_9$.

Solution: From *Property 3* of arithmetic progressions, we have:

a) $a_4 + a_8 = a_6 + a_6$ ($4 + 8 = 6 + 6$)

$$42 = 2a_6$$

$$a_6 = 21;$$

b) $a_4 + a_8 = a_3 + a_9$ ($4 + 8 = 3 + 9$)

$$42 = a_3 + a_9$$

$$a_3 + a_9 = 42.$$

EXAMPLE 6 For an arithmetic progression, we know that $a_8 = 12$. Find:

a) $a_7 + a_9$;

b) $a_2 + a_{14}$.

Solution:

a) $a_8 + a_8 = a_7 + a_9$ ($8 + 8 = 7 + 9$)

$$12 + 12 = a_7 + a_9$$

$$a_7 + a_9 = 24;$$

b) $a_8 + a_8 = a_2 + a_{14}$ ($8 + 8 = 2 + 14$)

$$12 + 12 = a_2 + a_{14}$$

$$a_2 + a_{14} = 24.$$

EXERCISES For which value of x are the numbers:

1. $x-1, x+3$, and x^2-5 positive consecutive terms of an arithmetic progression?

2. $x-2, x+4$, and x^2-2 consecutive terms of an increasing arithmetic progression?

3. $5, x+5$, and x^2+2 consecutive terms of a decreasing arithmetic progression?

In an arithmetic progression, we know that:

4. $a_7 + a_{13} = 48$. Find: a) a_{10} ; b) $a_5 + a_{15}$.

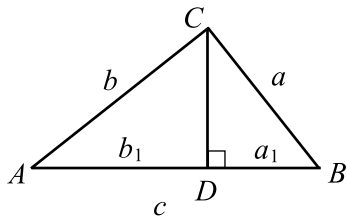
5. $a_2 + a_{10} = 24$.

Find: a) $a_5 + a_6 + a_7$; b) $a_4 + a_5 + a_6 + a_7 + a_8$.

6. $a_{12} = 64$. Find: a) $a_{10} + a_{14}$; b) $a_6 + a_{18}$.

T₃**Pythagorean Theorem**

The sum of the squares of the legs in a right triangle is equal to the square of the hypotenuse: $a^2 + b^2 = c^2$.

Proof:

We know that $\begin{cases} a^2 = a_1 \cdot c \\ b^2 = b_1 \cdot c \end{cases} +$

We add the two equalities and obtain:

$$\begin{aligned} a^2 + b^2 &= a_1 \cdot c + b_1 \cdot c = \\ &= c \cdot (a_1 + b_1) = c \cdot c = c^2. \end{aligned}$$

The **converse theorem** is also true:

T₄

If the sum of the squares of two sides in a triangle is equal to the square of the third side, then the triangle is right.

The two theorems can be formulated as one in the following way:

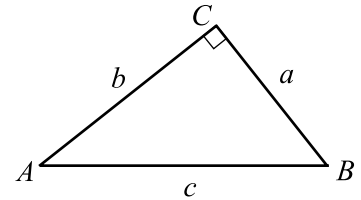
T₅

$\triangle ABC$ with sides $BC = a$, $CA = b$, and $AB = c$ is right with a right angle at vertex C if and only if $a^2 + b^2 = c^2$:

$$\triangle ABC, \angle C = 90^\circ \Leftrightarrow a^2 + b^2 = c^2.$$

EXAMPLE 1 Find the third side of $\triangle ABC$ ($\angle C = 90^\circ$) with legs a and b and hypotenuse c , provided:

- $a = 6$ and $b = 8$;
- $a = 5$ and $c = 13$;
- $b = \sqrt{7}$ and $c = 4$.

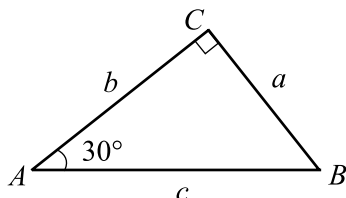
**Solution:**

$$\begin{aligned} \text{a) } a^2 + b^2 &= c^2 \\ 6^2 + 8^2 &= c^2 \\ 36 + 64 &= c^2 \\ c^2 &= 100 \\ c &= 10 \end{aligned}$$

$$\begin{aligned} \text{b) } a^2 + b^2 &= c^2 \\ 5^2 + b^2 &= 13^2 \\ 25 + b^2 &= 169 \\ b^2 &= 144 \\ b &= 12 \end{aligned}$$

$$\begin{aligned} \text{c) } a^2 + b^2 &= c^2 \\ a^2 + (\sqrt{7})^2 &= 4^2 \\ a^2 + 7 &= 16 \\ a^2 &= 9 \\ a &= 3 \end{aligned}$$

EXAMPLE 2 Find the legs of a right $\triangle ABC$ ($\angle C = 90^\circ$) with $\angle BAC = 30^\circ$ and $AB = 8$ cm.

Solution:

$$1. a = \frac{c}{2} = \frac{8}{2} = 4 \text{ cm (leg opposite } 30^\circ)$$

$$2. a^2 + b^2 = c^2 \Rightarrow 4^2 + b^2 = 8^2 \Rightarrow$$

$$b^2 = 8^2 - 4^2 = 48 \Rightarrow b = \sqrt{48} = 4\sqrt{3} \text{ cm}$$

EXAMPLE 3 Is $\triangle ABC$ right, provided its sides are:

a) $a = \sqrt{5}, b = 2, c = 3$;

b) $a = 3, b = 4, c = 5$;

c) $a = \sqrt{17}, b = 2\sqrt{2}, c = 5$?

Solution:

$$\begin{aligned} \text{a) } a^2 + b^2 &= \\ &= (\sqrt{5})^2 + 2^2 = \\ &= 5 + 4 = 9 \\ c^2 &= 3^2 = 9 \\ \Rightarrow a^2 + b^2 &= c^2 \\ \Rightarrow \angle C &= 90^\circ \end{aligned}$$

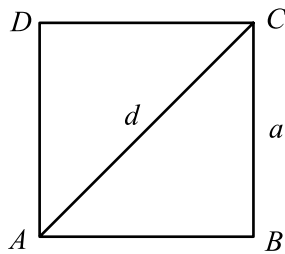
$$\begin{aligned} \text{b) } a^2 + b^2 &= \\ &= 3^2 + 4^2 = \\ &= 9 + 16 = 25 \\ c^2 &= 5^2 = 25 \\ \Rightarrow a^2 + b^2 &= c^2 \\ \Rightarrow \angle C &= 90^\circ \end{aligned}$$

$$\begin{aligned} \text{c) } a^2 + b^2 &= \\ &= (\sqrt{17})^2 + (2\sqrt{2})^2 = \\ &= 17 + 8 = 25 \\ c^2 &= 5^2 = 25 \\ \Rightarrow a^2 + b^2 &= c^2 \\ \Rightarrow \angle C &= 90^\circ \end{aligned}$$

Answer: Yes, $\triangle ABC$ is right.

EXAMPLE 4 Find the diagonal of a square with side a .

Solution:

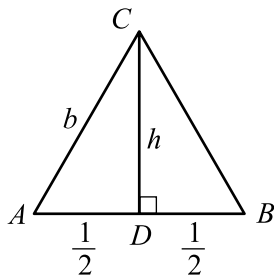


We apply the Pythagorean Theorem to $\triangle ABC$:

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ d^2 &= a^2 + a^2 \\ d^2 &= 2a^2 \\ d &= a\sqrt{2} \end{aligned}$$

EXAMPLE 5 Find the altitude of an equilateral triangle with side b .

Solution:



1. $AD = BD = \frac{b}{2}$ (CD altitude, median)

2. We apply the Pythagorean Theorem to $\triangle ACD$:

$$\begin{aligned} AC^2 &= AD^2 + CD^2 \\ b^2 &= \left(\frac{b}{2}\right)^2 + h^2 \\ h^2 &= b^2 - \frac{b^2}{4} \Rightarrow h^2 = \frac{3b^2}{4} \Rightarrow h = \frac{b\sqrt{3}}{2} \end{aligned}$$

- EXERCISES**
- Find the third side of $\triangle ABC$ ($\angle C = 90^\circ$) with legs a and b and hypotenuse c if:
 - $a = 1$ and $b = 1$;
 - $a = 2$ and $c = 5$;
 - $a = 5$ and $b = 12$;
 - $b = 5$ and $c = 13$.
 - Find the legs of $\triangle ABC$ with $\angle C = 90^\circ$, $\angle BAC = 60^\circ$, and $AB = 10$ cm.
 - Find the legs of an isosceles right triangle with hypotenuse 4 cm.
 - Check if the triangle is right, provided its sides have the following lengths:
 - $a = 15, b = 20$, and $c = 25$;
 - $a = 4, b = 8$, and $c = 4\sqrt{5}$;
 - $a = 7, b = 24$, and $c = 25$;
 - $a = 6, b = 8$, and $c = 12$.